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Entanglement and Properties of Composite Quantum Systems: A Conceptual and Mathematical Analysis

GianCarlo Ghirardi,[1,2] Luca Marinatto,[1] and Tullio Weber[1]

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Various topics concerning the entanglement of composite quantum systems are considered with particular emphasis concerning the strict relations of such a problem with the one of attributing objective properties to the constituents. Most of the paper deals with composite systems in pure states. After a detailed discussion and a precise formal analysis of the case of systems of distinguishable particles, the problems of entanglement and the one of the properties of subsystems of systems of identical particles are thoroughly discussed. This part is the most interesting and new and it focuses in all details various subtle questions which have never been adequately discussed in the literature. Some inappropriate assertions which appeared in recent papers are analyzed. The relations of the main subject of the paper with the nonlocal aspects of quantum mechanics, as well as with the possibility of deriving Bell's inequality are also considered.

KEY WORDS: Entanglement; identical particles.

1. INTRODUCTION

One of the crucial points of any theory aiming to account for natural phenomena concerns the possibility of identifying the properties objectively possessed by individual physical systems and/or by their constituents. Such a problem acquires a completely different status within different theoretical schemes, typically in the classical and quantum cases. First of all, quantum mechanics, if the completeness assumption is made, requires a radical change of attitude about the problem of attributing objective properties to physical systems due to its fundamentally probabilistic character. Secondly,

[1] Department of Theoretical Physics of the University of Trieste, and Istituto Nazionale di Fisica Nucleare, Sezione di Trieste; e-mail: ghirardi@ts.infn.it, marinatto@ts.infn.it, and weber@ts.infn.it

[2] International Centre for Theoretical Physics, Trieste, Italy.

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and even more important for our analysis, it gives rise to specific and puzzling situations concerning the properties of the constituents of a composite system due to its peculiar feature, Entanglement—the direct English translation of the original German form *Verschrankung* used by Schrödinger[1] —which Schrödinger himself considered "*the characteristic trait of Quantum Mechanics, the one that enforces its entire departure from classical line of thoughts.*"

This paper is devoted to analyze quantum entanglement, to characterize it in a clear and rigorous way, to derive various new theorems allowing to identify its occurrence and to point out some misleading and/or erroneous arguments about it which can be found in the literature. Particular emphasis is given to the conceptual and formal changes which are necessary to deal in a logically correct way with the problem of entanglement of systems involving identical constituents.

The paper is divided in four parts and is organized as follows: in Part I the possible probabilistic features of various (classical and quantum) theories are discussed with the purpose of illustrating the interplay between the epistemic and nonepistemic aspects of the description of natural processes and of characterizing the different types of "states" which one has to take into account according to the information he has about the system. The problem of the attribution of objective properties to individual physical systems in the classical and quantum cases is also discussed. The rest of the paper is entirely devoted to quantum systems. In Parts II and III the most extended and relevant, we investigate the implications of entanglement of composite quantum systems concerning properties, by confining our considerations to systems in pure states, or, equivalently, to homogeneous quantum ensembles. Part II deals with the case of distinguishable constituents, while Part III is entirely devoted to present a detailed and original analysis of the case involving identical constituents. In Part IV we take briefly into account the non-pure cases, or, equivalently, the non-homogeneous quantum ensembles, and we discuss some relevant questions connected with quantum nonlocality and Bell's inequality.

PART I. PROBABILITIES AND PROPERTIES

2. PHYSICAL THEORIES AND THEIR PROBABILISTIC FEATURES

Any theoretical scheme aiming to account for natural processes describes the state of individual physical systems and the physically observable quantities by appropriate mathematical entities. The scheme must contemplate rules mirroring the crucial steps of the unfolding of a process: the preparation of the system, its evolution and the recipes by which one can make predictions concerning the outcomes of prospective measurement processes on the system. All the just mentioned stages can exhibit deterministic or probabilistic aspects. In the case in which one has to resort (for various reasons which we will analyze in what follows) to a probabilistic description, one is naturally led to raise the conceptually relevant question of the precise status assigned to probabilities within the scheme, in particular whether they have an epistemic or nonepistemic character. Obviously, answering such a question requires a specific analysis of the logical and formal structure of the theory. In fact it is quite easy to exhibit physically equivalent theoretical schemes (one of the best known examples being Bohmian Mechanics and Standard Quantum Mechanics) whose probabilities have a completely different conceptual status.

Let us therefore start by discussing the notion of *state of an individual physical system* within the hypothetical theory under consideration. The crucial point, from a conceptual point of view, consists in identifying which is the most accurate characterization that the theory allows concerning the situation of an individual physical system. By taking this attitude, we are disregarding (for the moment) the unavoidable difficulties one meets in actually preparing a system in such a way that its physical situation corresponds to the just mentioned most accurate specification allowed by the formalism, and/or in knowing precisely its situation at a given time. For our present purposes we assume that such a "most accurate preparation" or "most exhaustive knowledge" is, in principle, possible. Such a characterization is expressed by mathematical entities which we will denote as the **States** (with capital S) of the theory. As the reader certainly knows, the problem we are facing is strictly related to the so called assumption of completeness of the theoretical framework: such an assumption amounts simply to accept that no specification more precise than the one given by the **States** is possible.

One can immediately exhibit some elementary examples of what we have in mind. For instance, the **States** of a system of N point particles within Newtonian mechanics are the points P of the $6N$-dimensional phase space of the system. Similarly, non-relativistic quantum mechanics with the completeness assumption asserts that the **States** of a system of N spinless particles are the state vectors of the associated Hilbert space, i.e., the square integrable functions $\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_N)$ of the $3N$ coordinates of the particles.

As already remarked it can very well happen that one is not able to prepare a system in a precise **State** or to have a precise knowledge of it. This impossibility may derive from practical limitations but it can also occur for reasons inherent to the theory itself. When we do not know the

State, but we still have some control on the preparation or some meaningful information about the physical situation, we will speak of the **state** (with lower case s) of the physical system.[3] Once more simple cases can be mentioned: for a mechanical classical system we can (in practice) know at most its **state** and never its **State**, this fact being due to the practical impossibility of identifying with infinite precision the point in phase space characterizing the precise physical situation of the system. Within Bohmian Mechanics, in spite of the fact that the most accurate specification of an individual physical system is given by the combined assignment of its wave function $\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_N)$ plus the positions $(\mathbf{r}_1,\ldots,\mathbf{r}_N)$ of its particles, it is usually assumed that while we can prepare a system in any chosen state vector, there is no possibility of controlling or knowing the positions of the particles in more detail than is conveyed by the quantum position probabilities.[4] We remark that, as is well known, if one could control the positions (which are often called—absurdly, according to Bell—the hidden variables of the theory) one would be able to falsify quantum mechanics as well as achieve superluminal signaling.

Having clarified this point, let us come back to our general theoretical scheme and let us confine, for the moment, our considerations to the case in which our physical system is in a precise **State**. As already stated, a satisfactory theory must contain some recipe (usually an evolution equation) allowing to deduce from the knowlegde of the **State** $S(0)$ at the initial time, some meaningful information about the physical situation at later times. Once more, the evolution may be deterministic or stochastic; in the first case it is a mapping of the set of the **States** into (or onto) itself, in the second case it is a mapping from the set of **States** to the set of **states**. Since we are interested in discussing the cases of standard quantum mechanics and of classical mechanics, we will assume that the evolution is perfectly deterministic and reversible, i.e., it is an injective and surjective mapping of the set of the **States** onto itself.

At this point we are led to analyze the last and essential feature of the theory, i.e., its allowing to make predictions about the outcomes of prospective measurements of physical observables. The question should be clear: we assume that we know the **State** $S(t)$ (i.e., the most accurate specification which the theory makes legitimate concerning an individual physical system) at a given time t, and we are interested in what the theory tells us about the outcomes of measurement procedures concerning all conceivable observable quantities at the considered time. Once more the predictions of

[3] Typically a state can be identified with a probability measure on the ensemble of States.

[4] For a detailed discussion about the conceptual status of this assumption see the analysis of ref. 2.

the theory can have a deterministic or a probabilistic character, the two paradigmatic cases being classical and quantum mechanics.

In classical mechanics the observable quantities are functions of the **State**, i.e., of the point in phase space associated to the precise physical situation we are considering. Let us denote as $F(\mathbf{r}_i, \mathbf{p}_i)$ a generic observable. Such a quantity takes a precise value at time t, which is simply given by $F(\mathbf{r}_i(t), \mathbf{p}_i(t))$. Accordingly, classical mechanics is a deterministic theory at its fundamental level, i.e., when analyzed in terms of its **States**. Obviously, probabilities can enter into play also within such a theory; as already mentioned this happens when we are not dealing with the **States** but with the **states** of the theory. However, the need to pass from the **States** to the **states** corresponds, for the considered case, to a lack of information about the system with respect to the one that the theory considers in principle possible: this allows to conclude that the probabilities of classical mechanics (and in particular the practically unavoidable ones of classical statistical mechanics) have an epistemic status, i.e., they are due to our ignorance about the physical system under consideration.

The situation in quantum mechanics is quite different. First of all, even when we deal with the **States** (i.e., with the so called pure cases in which we know precisely the state vector of the system) the theory attaches (in general) probabilities different from 0 and 1 to the outcomes of measurements concerning almost all physical observables F, which, as is well known, are represented by self-adjoint operators $\hat{F}$. This means that, when the completeness assumption is made, quantum probabilities have a nonepistemic status. However, it is useful to remark that for any given state vector (a **State** in our language) there is always one (actually infinitely many) self-adjoint operator such that the considered state vector is an eigenstate of it belonging to an appropriate eigenvalue. For such an observable the theory attaches probability 1 to the outcome corresponding to the eigenvalue in a measurement of the related observable, so that one can predict with certainty the outcome. Even more: the formalism tells us that for a system in a pure state there are complete sets of commuting observables such that the state vector is a simultaneous eigenstate of all of them.[5]

When, within a quantum scheme, one passes from the consideration of the **States** to that of the **states**, the situation becomes richer and deserves further comments. In fact, two conceptually different situations can occur,

[5] To be more precise, given any complete set of commuting observables such that all its members commute with the projection operator on the one-dimensional manifold spanned by the state vector, the state vector itself is a common eigenvector of the considered complete set of observables. Accordingly, in the case under consideration there are various complete sets of observables such that the theory attaches probability 1 to a corresponding specific set of eigenvalues for each set.

depending on the information one has about the system (or the inhomogeneous ensemble) associated to a **state**.

In the first case one knows the precise probabilities p_i of the system being in the **State** $|\phi_i\rangle$, or, equivalently, the precise fractions p_i of the members of the ensemble which are in the considered state $|\phi_i\rangle$. Then an interplay between epistemic and nonepistemic probabilities occurs: if we are interested in a specific observable G, in order to evaluate the probability of getting the outcome g_k in a measurement we have to argue as follows. There is an epistemic probability p_i that my system (or the individual which is picked up from the ensemble) is described by the pure state $|\phi_i\rangle$, and such a state attaches (in general) a nonepistemic probability, let us say $\pi_{i,k}$ to the outcome g_k. It has to be remarked that if one takes such an attitude (i.e., he knows that the system can be only in one of the state vectors $|\phi_i\rangle$ but he does not know precisely which one) it remains true that for each such state there are precise observables (different for different values of the index i) which have probability one of giving appropriate outcomes. This remark is relevant for the problem of identifying the properties which can be considered as objectively possessed by individual physical systems. As it is well known, the consideration of the statistical operator ρ (a trace class, trace one, semipositive definite operator) is the mathematically appropriate entity to deal with the **states** of the system (or of the ensemble). For the case considered above, it has the expression[6] $\rho = \sum_r p_r |\phi_r\rangle\langle\phi_r|$.

In the second case one knows the statistical operator but one is ignorant about the precise composition of the ensemble associated to the **state** under consideration. We stress the conceptual relevance of making the just mentioned distinction between the two above cases. It derives from the precise formal fact that, while in classical mechanics non-pure states (i.e., **states** in our language) are in one-to-one correspondence with statistical ensembles, in quantum mechanics this is by no means true; actually the correspondence between statistical ensembles and statistical operators is infinitely many to one. To give just an elementary example, we can mention the case of an ensemble which is the union of two subensembles which are pure cases associated to the orthogonal states $|\varphi_1\rangle$ and $|\varphi_2\rangle$, with weights p_1 and p_2, respectively, and suppose $p_1 > p_2$. The statistical operator can be written

$$\begin{aligned}\rho &= p_1 |\varphi_1\rangle\langle\varphi_1| + p_2 |\varphi_2\rangle\langle\varphi_2| \\ &\equiv p_2[|\varphi_1\rangle\langle\varphi_1| + |\varphi_2\rangle\langle\varphi_2|] + (p_1 - p_2)|\varphi_1\rangle\langle\varphi_1| \end{aligned} \tag{2.1}$$

[6] It goes without saying that also in the case of a **State** one can resort to the statistical operator formalism. In the considered case the statistical operator turns out to be the projection operator on the one-dimensional manifold spanned by the state vector and, as such, it satisfies the condition $\rho^2 = \rho$.

One can then notice that in the second expression for ρ the number p_2 multiplies the projection operator on the two dimensional manifold spanned by the set $\{|\varphi_1\rangle, |\varphi_2\rangle\}$. Such a projection can be written in terms of any pair of orthogonal vectors $|\mu_1\rangle, |\mu_2\rangle$, spanning the same manifold, so that ρ can also be written as:

$$\rho = (p_1 - p_2)|\varphi_1\rangle\langle\varphi_1| + p_2\,|\mu_1\rangle\langle\mu_1| + p_2\,|\mu_2\rangle\langle\mu_2| \tag{2.2}$$

Equation (2.2) shows that ρ is also the statistical operator describing an ensemble which is the union of three pure subensembles associated to the nonorthogonal states $\{|\varphi_1\rangle, |\mu_1\rangle, |\mu_2\rangle\}$ with the indicated weights. Since the state vectors $|\mu_1\rangle, |\mu_2\rangle$ are, in general, eigenstates of observables different and incompatible with those having $|\varphi_2\rangle$ as an eigenvector, the observables that have definite values when one member of the ensemble is chosen become ambiguous when we specify only the statistical operator and not the actual composition of the ensemble. Obviously, the probabilities that the theory attaches to the outcomes of all conceivable measurement processes coincide for all ensembles associated to the same **state** ρ, but from a conceptual point of view there is a subtle difference between the two cases, which should be clear to the reader and which we will reconsider in Section 3.

The considerations of this section, when reference is made to the two cases of interest for us, i.e., Classical Mechanics and Quantum Mechanics, can be summarized as follows:

Classical Mechanics:

State	Point of the phase space (q_i, p_i)	Determinism
state	Probability measure on the phase space $\rho\,(q_i, p_i)$	Epistemic Probabilities

Quantum Mechanics:

State	State Vector $|\psi\rangle$	Nonepistemic Probabilities
state	Statistical operator $\rho = \sum_i p_i\,|\psi_i\rangle\langle\psi_i|$	Epistemic and Nonepistemic Probabilities

As discussed above, within quantum mechanics it is useful to keep in mind that for a given **state** we can still have different information about the ensemble associated to it, according whether we know the composition (weights and state vectors) of the ensemble itself in terms of its pure subensembles or we know nothing besides the statistical operator.

3. PROPERTIES OF INDIVIDUAL PHYSICAL SYSTEMS

In this section we tackle the problem of attributing properties to individual physical systems. In order to come to the most interesting point of our analysis, i.e., to discuss the specific problems which arise in connection with this matter in the case of composite quantum systems in entangled states, it is appropriate to reconsider briefly the case of Classical Mechanics. Within such a theory, as already stated, all conceivable observables, both referring to the whole system as well as to all its subsystems, are functions of the positions and momenta of the particles, so that, when one knows the **State** of the system, i.e., the phase space point associated to it, one knows also the precise values of all physical observables. We can claim that within Classical Mechanics all properties are objectively possessed, in the precise sense that the measurement of any given observable simply reveals the pre-existing value possessed by the observable.[7] It goes without saying that if we lack the complete information about the system, then we can make statements only concerning the (epistemic) probabilities that it possesses precise properties. Nevertheless, it remains true that any individual system (and its subsystems) has all conceivable properties, in spite of the fact that we can be ignorant about them.[8]

As everybody knows, the situation is quite different in quantum mechanics due to the nonabelian structure of the set of the observables. Accordingly, as already discussed, the theory, in general, consents to make only nonepistemic probabilistic predictions about the outcomes of measurement processes even when the **State** of the system is known. However, in such a case, there are always complete sets of commuting observables such that the theory attaches probability one to a precise outcome in a measurement process of any one of them. It is then natural to assume (as we will do) that when we can make certain (i.e., with probability one) predictions about the outcomes, the system possesses objectively the property, or element of physical reality "such an observable has such a value,"

[7] Here we have tacitly assumed that one can perform ideally faithful measurements, i.e., measurements which reveal precisely the value of the quantities they are devised to measure. In what follows we will also make the corresponding idealized assumption for the quantum case, i.e., that if the state vector is an eigenstate of an observable, its measurement will yield with certainty the associated eigenvalue.

[8] Stated differently, claims of the kind "the energy of this particle has this specific value" have truth values, i.e., they are definitely either true or false.

independently of our decision to measure it. Here we have used the expression *objective properties* and *elements of reality* with the same meaning that Einstein[(3)] gave them in the analysis of the EPR paradox:

> *If, without in any way disturbing a system, we can predict with certainty (i.e., with a probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.*

When one takes into account the just outlined situation, one can concisely express the lesson that quantum mechanics has taught us, by stating that within such a theory one cannot consider (even in principle) an individual physical system as possessing objectively too many properties. Some of them can be legitimately considered as *actual*, all the other have the ontological status of *potentialities*. At any rate, according to the remarks of the previous section, a system in a pure state always has complete sets of definite and objective properties.

Obviously, when we have not the most accurate knowledge of the physical situation of the system under consideration, i.e., when we know only its **state**, then, in general, we can at most make epistemic probabilistic statements even about the limited set of properties that the system might possess. As discussed in the previous section we can, in principle, have a different knowledge of the state of the system according whether we know the precise composition of the ensemble to which it belongs or only the statistical operator associated to it. The difference can be easily appreciated by considering a particular instance of the situation analyzed in the previous section concerning different ensembles associated to the same statistical operator. Suppose we have an unpolarized beam of spin $1/2$ particles and we are interested only in their spin properties. The statistical operator corresponding to it is $(1/2)\, I$ (I being the identity operator). However, such a state can describe, e.g., an ensemble of particles with uniform distribution of their spins over all directions, or an ensemble obtained by putting together an equal number of particles in the eigenstates associated to the eigenvalue $+1$ and -1 of the observable σ_z, respectively. While in the second case the statement "each particle of the ensemble has surely either the spin *up* or *down* along the z-axis" is legitimate and true independently of any measurement being actually performed, it is certainly illegitimate in the first one.[9]

[9] Obviously, in both cases, if one subjects all particles to a measurement of their z-spin component, he will get almost in $1/2$ of the cases the outcome *up* and in $1/2$ of the cases the outcome *down*. However, here we are not making exclusive reference to the outcomes, but to the possibility of considering a property as objectively (i.e., independently of our decision to perform a measurement) possessed. From this point of view the two cases are radically different.

Up to this point we have confined our attention to quantum systems considered as a whole. However, as already mentioned, the phenomenon of quantum entanglement makes the situation much more puzzling when consideration is given to composite quantum systems and one raises the problem of the properties of their constituents. As we will see, in such a case it is very common to meet situations (most of which arise as a consequence of the interactions between the constituents) in which the constituents themselves do not possess *any property whatsoever*. This is a new feature which compels us to face a quite peculiar state of affairs: not only must one limit drastically the actual properties of physical systems (being in any case true that the system as a whole always has some properties), but one is forced also to accept that the parts of a composite system can have no property at all. Only the entire system, even if its parts are far apart and noninteracting, has some properties, while its parts have only *potentialities* and no *actualities*. In this way the quantum picture of the universe as an "unbroken whole," or as "undivided," emerges.

Quantum entanglement has played a central role in the historical development of quantum mechanics, in particular since it has compelled the scientific community to face the essentially nonlocal features of natural processes. Nowadays, entangled states have become the essential ingredients of all processes involving teleportation and quantum cryptography and constitute an important tool for implementing efficient quantum algorithms. This explains why a great deal of efforts has been spent by theorists during the last years in trying to characterize the very nature and properties of entanglement, and this is also the reason which motivates our attempt to deepen some questions about these matters.

PART II. ENTANGLEMENT AND PROPERTIES OF QUANTUM SYSTEMS OF DISTINGUISHABLE PARTICLES IN PURE STATES

As already stressed, the problem of attributing properties to the constituents of composite systems in entangled states is a rather delicate one. Here we will discuss this problem and we will derive a series of significant theorems with particular reference to systems of distinguishable particles in pure states.

4. ENTANGLEMENT OF TWO DISTINGUISHABLE PARTICLES

In this section we study the **Entanglement** between two distinguishable particles $\mathscr{S}_1$ and $\mathscr{S}_2$. Let us suppose that the two particles are parts of a larger quantum system $\mathscr{S} = \mathscr{S}_1 + \mathscr{S}_2$, whose associated Hilbert space $\mathscr{H}$

is the direct product of the Hilbert spaces of the single subsystems, $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$. As already stated, in this Part of the paper we will always assume that the composite quantum system $\mathscr{S}$ is described by a state vector $|\psi(1,2)\rangle \in \mathscr{H}$ or, in a totally equivalent manner, by a pure density operator $\rho = |\psi(1,2)\rangle\langle\psi(1,2)|$.

4.1. General Definition and Theorems

Let us start by characterizing a non-entangled (a separable) composite system by making explicit reference to the fact that one of its two constituent subsystems possesses complete sets of properties (as we will see this in turn implies that the same is true for the other constituent):

Definition 4.1. The system $\mathscr{S}_1$, subsystem of a composite quantum system $\mathscr{S} = \mathscr{S}_1 + \mathscr{S}_2$ described by the pure density operator ρ, is **non-entangled** with subsystem $\mathscr{S}_2$ if there exists a projection operator $P^{(1)}$ onto a one-dimensional manifold of $\mathscr{H}_1$ such that:

$$Tr^{(1+2)}[P^{(1)} \otimes I^{(2)}\rho] = 1 \tag{4.1}$$

The fact that in the case of non-entangled states it is possible to consider each one of the constituents as possessing complete sets of well definite physical properties, independently of the existence of the other part, follows directly from the following theorem:

Theorem 4.1. If consideration is given to a composite quantum system $\mathscr{S} = \mathscr{S}_1 + \mathscr{S}_2$ described by the pure state vector $|\psi(1,2)\rangle$ (or, equivalently by the pure density operator $\rho = |\psi(1,2)\rangle\langle\psi(1,2)|$) of $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$, each of the following three conditions is necessary and sufficient in order that subsystem $\mathscr{S}_1$ is non-entangled with subsystem $\mathscr{S}_2$:

1. there exists a projection operator $P^{(1)}$ onto a one-dimensional manifold of $\mathscr{H}_1$ such that $Tr^{(1+2)}[P^{(1)} \otimes I^{(2)}\rho] = 1$;

2. the reduced statistical operator $\rho^{(1)} = Tr^{(2)}[\rho]$ of subsystem $\mathscr{S}_1$ is a projection operator onto a one-dimensional manifold of $\mathscr{H}_1$;

3. the state vector $|\psi(1,2)\rangle$ is factorizable, i.e., there exist a state $|\phi(1)\rangle \in \mathscr{H}_1$ and a state $|\xi(2)\rangle \in \mathscr{H}_2$ such that $|\psi(1,2)\rangle = |\phi(1)\rangle \otimes |\xi(2)\rangle$.

Proof. If subsystem $\mathscr{S}_1$ is non-entangled with $\mathscr{S}_2$ then, according to Definition 4.1 condition 1 is satisfied, i.e., $Tr^{(1+2)}[P^{(1)} \otimes I^{(2)}\rho] = Tr^{(1)}[P^{(1)}\rho^{(1)}] = 1$. Since $P^{(1)}$ projects onto a one-dimensional manifold and $\rho^{(1)}$ is a statistical operator (i.e., a trace-class, trace one, semipositive definite and hermitian operator), the last equality implies $\rho^{(1)} = P^{(1)}$, i.e., $\rho^{(1)}$ is a projection operator onto a one-dimensional manifold.

If $\rho^{(1)}$ is a projection operator onto a one-dimensional manifold (condition 2) then it is useful to resort to von Neumann's biorthonormal decomposition of the state $|\psi(1,2)\rangle$ in terms of states of $\mathscr{H}_1$ and $\mathscr{H}_2$ (if there is any accidental degeneracy we can dispose of it as we want):

$$|\psi(1,2)\rangle = \sum_k \pi_k \, |\phi_k(1)\rangle \otimes |\xi_k(2)\rangle \tag{4.2}$$

where the π_k are real and positive numbers satisfying $\sum_k \pi_k^2 = 1$. Equation (4.2) implies:

$$Tr^{(2)}[|\psi(1,2)\rangle\langle\psi(1,2)|] = \sum_k \pi_k^2 \, |\phi_k(1)\rangle\langle\phi_k(1)| \tag{4.3}$$

The r.h.s. of this equation, due to the orthogonality of the states $|\phi_k(1)\rangle$, can coincide with a projection operator onto a one-dimensional manifold iff the sum in (4.3) contains only one term, let us say the first one, the corresponding coefficient π_1 taking the value 1. Accordingly, from (4.2) we get:

$$|\psi(1,2)\rangle = |\phi_1(1)\rangle \otimes |\xi_1(2)\rangle \tag{4.4}$$

i.e., $|\psi(1,2)\rangle$ is factorized.

Finally, if $|\psi(1,2)\rangle$ is factorized as in (4.4), then the one-dimensional projection operator $P^{(1)} = |\phi_1(1)\rangle\langle\phi_1(1)|$ satisfies:

$$Tr^{(1+2)}[P^{(1)} \otimes I^{(2)} \rho] = 1 \tag{4.5}$$

and subsystem S_1 is not entangled with S_2. ∎

4.2. Entanglement and Properties of Two Distinguishable Particles

The analysis of the previous subsection allows us to conclude that if a quantum system composed of two subsystems is non-entangled, the states of subsystems $\mathscr{S}_1$ and $\mathscr{S}_2$ are completely specified, in the sense that it is possible to associate to each of them a unique and well-defined state vector. According to our previous discussion, the individual subsystems can therefore be thought of as having complete sets of definite and objective properties of their own.

We pass now to analyse the case of composite systems of two subsystems in entangled states. According to Theorem 4.1, the reduced density operator of each subsystem is not a projection operator onto a one dimensional manifold. It is then useful to analyse whether there exist projection operators on manifolds of dimension greater than or equal to 2 of $\mathscr{H}_1$, satisfying condition (4.1). As shown by the following theorem, there is a strict relation between such projection operators and the range $\mathscr{R}[\rho^{(1)}]$ of the reduced statistical operator $\rho^{(1)}$:

Theorem 4.2. A necessary and sufficient condition for the projection operator $P^{(1)}_{\mathscr{M}_1}$ onto the linear manifold $\mathscr{M}_1$ of $\mathscr{H}_1$ to satisfy the two following conditions:

1. $Tr^{(1)}[P^{(1)}_{\mathscr{M}_1}\rho^{(1)}] = 1$;
2. there is no projection operator $\tilde{P}^{(1)}$ of $\mathscr{H}_1$ satisfying the conditions $\tilde{P}^{(1)} < P^{(1)}_{\mathscr{M}_1}$ (i.e., it projects onto a proper submanifold $\mathscr{N}_1$ of $\mathscr{M}_1$) and $Tr^{(1)}[\tilde{P}^{(1)}\rho^{(1)}] = 1$,

is that the range $\mathscr{R}[\rho^{(1)}]$ of the reduced statistical operator $\rho^{(1)}$ coincides with $\mathscr{M}_1$.

Proof. If $\mathscr{R}[\rho^{(1)}] = \mathscr{M}_1$, then $P^{(1)}_{\mathscr{M}_1}\rho^{(1)} = \rho^{(1)}$ implying that $Tr^{(1)}[P^{(1)}_{\mathscr{M}_1}\rho^{(1)}] = Tr^{(1)}[\rho^{(1)}] = 1$, while any projection operator $\tilde{P}^{(1)}$ on a proper submanifold of $\mathscr{M}_1$ satisfies $Tr^{(1)}[\tilde{P}^{(1)}\rho^{(1)}] < 1$.

Conversely if $Tr^{(1)}[P^{(1)}_{\mathscr{M}_1}\rho^{(1)}] = 1$ we consider the spectral decomposition of $\rho^{(1)}$, where only eigenvectors corresponding to non-zero eigenvalues appear:

$$\rho^{(1)} = \sum_i p_i\, |\varphi_i(1)\rangle\langle\varphi_i(1)|, \qquad \sum_i p_i = 1, \qquad p_i \neq 0 \ \forall i \tag{4.6}$$

We then have

$$\begin{aligned} Tr^{(1)}[P^{(1)}_{\mathscr{M}_1}\rho^{(1)}] &= \sum_i p_i\, Tr^{(1)}[P^{(1)}_{\mathscr{M}_1}|\varphi_i(1)\rangle\langle\varphi_i(1)|] \\ &= \sum_i p_i\, \|P^{(1)}_{\mathscr{M}_1}|\varphi_i(1)\rangle\|^2 = 1 \end{aligned} \tag{4.7}$$

implying, since $\sum_i p_i = 1$ and $\|P^{(1)}_{\mathscr{M}_1}\,|\varphi_i(1)\rangle\|^2 \leqslant 1$, that $P^{(1)}_{\mathscr{M}_1}\,|\varphi_i(1)\rangle = |\varphi_i(1)\rangle$.

The last equation shows that $P^{(1)}_{\mathscr{M}_1}$ leaves invariant $\mathscr{R}[\rho^{(1)}]$ so that $\mathscr{M}_1 \supseteq \mathscr{R}[\rho^{(1)}]$. If the equality sign holds we have proved the sufficiency. On the contrary if $\mathscr{M}_1 \supset \mathscr{R}[\rho^{(1)}]$, then the projection operator on the closed submanifold $\mathscr{R}[\rho^{(1)}]$ of $\mathscr{M}_1$ satisfies condition $Tr^{(1)}[P^{(1)}_{\mathscr{R}[\rho^{(1)}]}\rho^{(1)}] = 1$ contrary to the assumptions. We have therefore proved that $\mathscr{R}[\rho^{(1)}] = \mathscr{M}_1$. ▮

Let us analyze in detail the consequences of the above theorem by studying the following two cases concerning the range of the reduced statistical operator $\rho^{(1)}$:

1. $\mathscr{R}[\rho^{(1)}] = \mathscr{M}_1 \subset \mathscr{H}_1$,
2. $\mathscr{R}[\rho^{(1)}] = \mathscr{H}_1$,

As it has just been shown, in the first case (recall that we are considering the case in which the dimensionality of the manifold $\mathscr{M}_1$ is strictly greater than one) the projection operator $P^{(1)}_{\mathscr{M}_1}$ on $\mathscr{M}_1$ is such that $Tr^{(1)}[P^{(1)}_{\mathscr{M}_1}\rho^{(1)}] = 1$. Accordingly, given any self-adjoint operator $\Omega^{(1)}$ of $\mathscr{H}_1$ which commutes with $P^{(1)}_{\mathscr{M}_1}$, if consideration is given to the subset $\mathscr{B}$ (a Borel set) of its spectrum coinciding with the spectrum of its restriction $\Omega_R = P^{(1)}_{\mathscr{M}_1}\Omega^{(1)}P^{(1)}_{\mathscr{M}_1}$ to $\mathscr{M}_1$, we can state that subsystem $\mathscr{S}_1$ has the objective (in general unsharp) property that $\Omega^{(1)}$ has a value belonging to $\mathscr{B}$. In particular, all operators which have $\mathscr{M}_1$ as an eigenmanifold, have a precise objective value. Summarizing, even though in the considered case we cannot say that subsystem $\mathscr{S}_1$ has a complete set of properties by itself (i.e., objectively), it still has some sharp or unsharp properties associated to any observable which commutes with $P^{(1)}_{\mathscr{M}_1}$.

On the contrary, in the second of the above cases, i.e., the one in which $\mathscr{R}[\rho^{(1)}] = \mathscr{H}_1$, we have to face the puzzling implications of entanglement in their full generality. In fact, repetition of the argument we have just developed leads in a straightforward way, when $\mathscr{M}_1$ coincides with $\mathscr{H}_1$, to the conclusion that the only projection operator $P^{(1)}$ on $\mathscr{H}_1$ satisfying $Tr^{(1)}[P^{(1)}\rho^{(1)}] = 1$ is the identity operator $I^{(1)}$ on the Hilbert space $\mathscr{H}_1$. The physical meaning of this fact should be clear to the reader: it amounts to state that subsystem $\mathscr{S}_1$ does not possess objectively any sharp or unsharp property, i.e., that there is no self-adjoint operator for which one can claim with certainty that the outcome of its measurement will belong to any proper subset of its spectrum. The only certain but trivial statement[10] which is legitimate is that the outcome will belong to the spectrum. In the above situation we say that subsystem $\mathscr{S}_1$ is *totally entangled* with subsystem $\mathscr{S}_2$.

In the just considered case of total entanglement we have (appropriately) assumed the Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ to be infinite-dimensional. If

[10] Obviously, the theory attaches precise probabilities to the outcome belonging to any chosen Borel subset of the spectrum, but no one of such probabilities takes the value 0 or 1. Stated differently, in the considered case, we cannot speak of any (even quite unsharp) property as actual, all conceivable properties having the ontological status of potentialities.

one is interested in finite, let us say N-dimensional Hilbert spaces, then, besides the fact that the range of $\rho^{(1)}$ is the whole Hilbert space $\mathscr{H}_1$, it can also happen that $\rho^{(1)}$ itself is a multiple (by the factor $1/N$) of the identity operator. In such a case, the subsystem $\mathscr{S}_1$ not only does not possess any objective property, but it is characterized by the fact that the probabilities of giving any outcome in a measurement of any complete set of commuting observables are all equal to $1/N$. In a very precise sense one could state that the system has only potentialities and moreover that they are totally indefinite. Since in many processes involving quantum teleportation, quantum cryptography and in the studies about quantum information one often makes reference to finite-dimensional Hilbert spaces, and thus one can easily meet the just mentioned situation, a specific term has been introduced to deal with this state of affairs in which the properties of each subsystem are completely indefinite. Accordingly, the entangled states of two subsystems for which the reduced statistical operators are multiples of the identity are usually referred to as *maximally entangled states.*[11]

In order to clarify the two paradigmatic situations we have just analyzed, let us consider the simple case of a system composed by an electron and a positron, which we will label as particle 1 and particle 2 respectively.

Example 1. Let us suppose that the e^-e^+ system is described by the following state vector (with obvious meaning of the symbols):

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2}}[|\vec{n}\uparrow\rangle_1\,|\vec{n}\downarrow\rangle_2 - |\vec{n}\downarrow\rangle_1\,|\vec{n}\uparrow\rangle_2] \otimes |R\rangle_1\,|L\rangle_2 \tag{4.8}$$

where we have indicated with $|R\rangle$ and $|L\rangle$ two orthonormal states, whose coordinate representations are two specific square-integrable functions having compact disjoint supports at Right and Left, respectively. The reduced density operator describing the electron (obtained by taking the trace on the degrees of freedom of the second particle) acts on the infinite dimensional Hilbert space, $\mathscr{H}_1 = \mathscr{C}^2 \otimes \mathscr{L}(R^3)$ and has the following form:

$$\rho^{(1)} = \tfrac{1}{2}[|\vec{n}\uparrow\rangle_1\,{}_1\langle\vec{n}\uparrow| + |\vec{n}\downarrow\rangle_1\,{}_1\langle\vec{n}\downarrow|] \otimes |R\rangle_1\,{}_1\langle R| = \tfrac{1}{2}I \otimes |R\rangle_1\,{}_1\langle R| \tag{4.9}$$

Even though we cannot say anything about the value of the spin along any arbitrary direction $\vec{n}$, we can nevertheless state that the electron is, with

[11] In the infinite-dimensional case, to meet maximal entanglement one is compelled to enlarge the class of states and to resort to states which cannot be associated to bounded, trace class, trace one, semipositive definite operators. We will not analyze here this case.

certainty, inside the bounded right region R, and an analogous statement, i.e., that it is inside the bounded region L, can be made concerning the positron. Therefore the subsystems do not possess a complete set of properties with respect to both spin and position, but have at least the element of reality of being in definite spatial regions. This possibility of making claims about some properties is due to the fact that the range of the statistical operator of Eq. (4.9) is a proper submanifold of $\mathscr{H}_1$, i.e., the two dimensional manifold spanned by $|\vec{n}\uparrow\rangle_1\,|R\rangle_1$ and $|\vec{n}\downarrow\rangle_1\,|R\rangle_1$.

Example 2. In place of state (4.8) we consider now the following state vector for the e^-e^+ system:

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2}}\,[|\vec{n}\uparrow\rangle_1\,|\vec{n}\downarrow\rangle_2 - |\vec{n}\downarrow\rangle_1|\vec{n}\uparrow\rangle_2]$$

$$\otimes\left[\sum_i c_i|\varphi_i(1)\rangle\,|\theta_i(2)\rangle\right], \qquad c_i \neq 0\ \ \forall i \tag{4.10}$$

$\{|\varphi_i(1)\rangle\}$ and $\{|\theta_i(2)\rangle\}$ being two complete orthonormal sets of the Hilbert spaces $\mathscr{L}(R^3)$ associated to the spatial degrees of freedom of the constituents. The reduced density operator for the electron is:

$$\rho^{(1)} = Tr^{(2)}[|\psi(1,2)\rangle\langle\psi(1,2)|] = \tfrac{1}{2}I^{(1)} \otimes \sum_i |c_i|^2\,|\varphi_i(1)\rangle\langle\varphi_i(1)| \tag{4.11}$$

In Eq. (4.11), $I^{(1)}$ is the identity operator in the spin space of the electron. Since the range of $\rho^{(1)}$ is now the whole Hilbert space of the first particle, according to the previous discussion we cannot attribute any element of reality referring to any conceivable observable of the electron: it possesses only potential and no actual properties.

We hope to have succeeded in giving the appropriate emphasis to the remarkable peculiarities of *the most characteristic trait of quantum mechanics* and in having made clear that it compels us to accept that *the subsystems of a composite system may have no property at all which can be considered as objectively possessed.*

Summarizing, with reference to the range $\mathscr{R}[\rho^{(1)}]$ of the reduced statistical operator, we can conclude that:

- $\mathscr{R}[\rho^{(1)}]$ = a one-dimensional manifold $\Rightarrow$ subsystem $\mathscr{S}_1$ is *non-entangled* with $\mathscr{S}_2$ $\Rightarrow$ it possesses complete sets of objective properties, the same holding true for $\mathscr{S}_2$;

- $\mathscr{R}[\rho^{(1)}]$ = a proper submanifold of $\mathscr{H}_1$ of dimension greater than $1 \Rightarrow$ subsystem $\mathscr{S}_1$ is *partially entangled* with $\mathscr{S}_2 \Rightarrow$ it possesses some objective properties, however not a complete set of them;

- $\mathscr{R}[\rho^{(1)}] = \mathscr{H}_1 \Rightarrow$ subsystem $\mathscr{S}_1$ is *totally entangled* with $\mathscr{S}_2 \Rightarrow$ it does not possess any objective property whatsoever.

Note that in the second case it may very well happen that while $\mathscr{R}[\rho^{(1)}]$ is a proper submanifold of $\mathscr{H}_1$, $\mathscr{R}[\rho^{(2)}]$ coincides with $\mathscr{H}_2$. Analogously, in the last case it may happen that $\mathscr{R}[\rho^{(2)}]$ is a proper submanifold of $\mathscr{H}_2$.

4.3. Entanglement and Correlations

Another consequence of the entanglement of a composite quantum system is the occurrence of strict correlations between appropriate observables of the component subsystems, even when they are far apart and non-interacting. This is expressed by the following theorem:

Theorem 4.3. Subsystem $\mathscr{S}_1$ is non-entangled with subsystem $\mathscr{S}_2$ iff, given the pure state $|\psi(1,2)\rangle$ of the composite system, the following equation holds for any pair of observables $A(1)$ of $\mathscr{H}_1$ and $B(2)$ of $\mathscr{H}_2$ such that $|\psi(1,2)\rangle$ belongs to their domains:

$$\begin{aligned}&\langle\psi(1,2)|\, A(1)\otimes B(2)\,|\psi(1,2)\rangle\\&\quad= \langle\psi(1,2)|\, A(1)\otimes I^{(2)}\,|\psi(1,2)\rangle\langle\psi(1,2)|I^{(1)}\otimes B(2)|\,\psi(1,2)\rangle\end{aligned} \tag{4.12}$$

Note that Eq. (4.12) implies that no correlation exists between such pairs of observables.

Proof. If $\mathscr{S}_1$ is non-entangled with $\mathscr{S}_2$ then, according to Theorem 4.1, $|\psi(1,2)\rangle$ is factorized, from which Eq. (4.12) follows trivially.

Let us now assume that Eq. (4.12) is satisfied by all bounded operators $A(1)$ and $B(2)$. Given the state $|\psi(1,2)\rangle$ we consider its biorthonormal decomposition

$$|\psi(1,2)\rangle = \sum_i p_i\,|\varphi_i(1)\rangle\,|\theta_i(2)\rangle \tag{4.13}$$

Now we choose:

$$A(1) = |\varphi_r(1)\rangle\langle\varphi_r(1)| \qquad B(2) = |\theta_r(2)\rangle\langle\theta_r(2)| \tag{4.14}$$

and we impose Eq. (4.12) to hold for the considered state and the chosen observables. Since:

$$\begin{aligned} \langle\psi(1,2)|\, A(1)\, B(2)\, |\psi(1,2)\rangle &= p_r^2 \\ \langle\psi(1,2)|\, A(1)\, |\psi(1,2)\rangle\langle\psi(1,2)|\, B(2)\, |\psi(1,2)\rangle &= p_r^4 \end{aligned} \tag{4.15}$$

the request (4.12) implies that $p_r = 1$ or that $p_r = 0$ for any r, which, by taking into account that $\sum_i p_i^2 = 1$ shows that only one term occurs in Eq. (4.13), i.e.,

$$|\psi(1,2)\rangle = |\varphi_k(1)\rangle\, |\theta_k(2)\rangle \tag{4.16}$$

for an appropriate k. We have thus proved that if the state is not factorized there is at least a pair of observables for which Eq. (4.12) is not satisfied. ▮

The two observables appearing in Eq. (4.12) being completely arbitrary, the previous theorem holds also for projection operators: with such a choice one sees that in the case of entanglement the joint probabilities $\langle\psi(1,2)|\, P^{(1)} \otimes P^{(2)}\, |\psi(1,2)\rangle$ for outcomes of independent measurement processes performed on both subsystems cannot be expressed, in general, as the product of the probabilities for the two outcomes. Now, within quantum mechanics with the completeness assumption, it is easy to prove[(4–7)] that the mere fact of performing a measurement on one of the two entangled systems cannot alter the probability of any given outcome for a measurement on the other. If one takes into account this fact, the just outlined situation implies that at least some probabilities for the outcomes of measurements on subsystem $\mathscr{S}_1$ depend on the outcomes of measurements performed on $\mathscr{S}_2$. Such a peculiar feature[12] displayed by systems in non-separable (i.e., non factorized) states is usually termed as *Outcome Dependence.*

As it is well known the simplest example of this curious characteristics of entangled quantum systems is represented by the paradigmatic case of the singlet state of two distinguishable spin 1/2-particles when one disregards the spatial degrees of freedom:

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2}}\,[|z\uparrow\rangle_1\, |z\downarrow\rangle_2 - |z\downarrow\rangle_1\, |z\uparrow\rangle_2] \tag{4.17}$$

The joint probability, $Pr(\sigma_{1z} = \uparrow, \sigma_{2z} = \downarrow)$, of finding the first particle with spin up and the second particle with spin down along z-direction,

[12] For a detailed discussion of this point, see refs. 8–11.

when two measurements along the same axis are actually performed, equals 1/2. On the other hand the probabilities $Pr(\sigma_{1_z} = \uparrow)$ and $Pr(\sigma_{2_z} = \downarrow)$ of getting the indicated outcomes, which are defined as:

$$
\begin{aligned}
Pr(\sigma_{1_z} = \uparrow) &= \sum_{y=\uparrow,\downarrow} Pr(\sigma_{1_z} = \uparrow, \sigma_{2_z} = y) \\
Pr(\sigma_{2_z} = \downarrow) &= \sum_{x=\uparrow,\downarrow} Pr(\sigma_{1_z} = x, \sigma_{2_z} = \downarrow)
\end{aligned}
\tag{4.18}
$$

are both equal to 1/2 so that:

$$
Pr(\sigma_{1_z} = \uparrow) \cdot Pr(\sigma_{2_z} = \downarrow) = \tfrac{1}{2} \cdot \tfrac{1}{2} \tag{4.19}
$$

Thus, in accordance with Theorem 4.3, the joint probability $Pr(\sigma_{1_z} = \uparrow, \sigma_{2_z} = \downarrow) = 1/2$ does not match the value (4.19), proving therefore the *Outcome Dependence* property displayed by non-separable states. This naturally leads to the conclusion that the probability distributions of the results of measurements on the two separate entangled subsystems in the singlet state are dependent from each other.

5. ENTANGLEMENT OF *N* DISTINGUISHABLE PARTICLES

In this section we extend the previous analysis to the case in which the quantum system under consideration is composed by more than two subsystems. This generalization is not trivial, since it requires to take into account all possible correlations between the component particles. In particular, it may happen that a group of particles, which can be entangled or non-entangled among themselves, is not entangled with the remaining ones. After having identified disentangled groups, one must repeat the analysis for the members of each group, up to the point in which he has grouped all the particles of the system (which here we assume to be all distinguishable from each other) into sets which are disentangled from each other, while no further decomposition is possible. Apart from this complication the problem can be tackled by following step by step the procedures we have used in the previous sections to deal with systems composed of two distinguishable constituents. The extreme case, as we will see, is that of a system in a state corresponding to completely non-entangled constituents. Since all arguments and proofs of the theorems we will present in this section can be obtained by repeating step by step those of the previous section for the case of two particles, we will limit ourselves to present the relevant definitions and theorems.

5.1. Entanglement between Subsets of the Constituents

We begin by considering the possibility of grouping the particles in two subsets which are non-entangled with each other. Obviously, the particles of each subset may be entangled among themselves. Let us consider a composite system of N distinguishable quantum particles, whose state vector $|\psi(1,..., N)\rangle$ belongs to the direct product of the single particle Hilbert spaces, $\mathscr{H} = \prod_i \otimes \mathscr{H}_i$, and let us adopt the following definition:

Definition 5.1. The subsystem $\mathscr{S}_{(1\cdots M)} = \mathscr{S}_1 + \cdots + \mathscr{S}_M$ of a composite quantum system $\mathscr{S} = \mathscr{S}_1 + \cdots + \mathscr{S}_M + \cdots + \mathscr{S}_N$ in the pure state $|\psi(1,..., N)\rangle$, is **non-entangled** with the subsystem $\mathscr{S}_{(M+1\cdots N)} = \mathscr{S}_{M+1} + \cdots + \mathscr{S}_N$ if there exists a one dimensional projection operator $P^{(1\cdots M)}$ of $\mathscr{H}_1 \otimes \cdots \otimes \mathscr{H}_M$, such that:

$$Tr^{(1+\cdots+N)}[P^{(1\cdots M)} \otimes I^{(M+1\cdots N)}|\psi(1,..., N)\rangle\langle\psi(1,..., N)|] = 1 \qquad (5.1)$$

This definition implies that we can attribute complete sets of objective properties (elements of reality) to at least the two subgroups of the particles we have indicated as $\mathscr{S}_{(1\cdots M)}$ and $\mathscr{S}_{(M+1\cdots N)}$. The natural generalization of the corresponding theorem for two particles is then easily derived by recalling that the biorthonormal decomposition holds in general for the direct product of two Hilbert spaces and by repeating step by step its proof:

Theorem 5.1. If consideration is given to a many-particle quantum system described by the pure state $|\psi(1,..., N)\rangle$ (or by the corresponding pure density operator) of the Hilbert space $\mathscr{H} = \prod_i \otimes \mathscr{H}_i$, each of the following three conditions is necessary and sufficient in order that subsystem $\mathscr{S}_{(1\cdots M)}$ is non-entangled with subsystem $\mathscr{S}_{(M+1\cdots N)}$:

1. there exists a projection operator $P^{(1\cdots M)}$ onto a one-dimensional manifold of $\mathscr{H}_1 \otimes \cdots \otimes \mathscr{H}_M$ such that $Tr^{(1+\cdots+N)}[P^{(1\cdots M)} \otimes I^{(M+1\cdots N)}\rho^{(1\cdots N)}] = 1$;

2. the reduced statistical operator $\rho^{(1\cdots M)} = Tr^{(M+1\cdots N)}[\rho^{(1\cdots N)}]$ of subsystem $\mathscr{S}_{(1\cdots M)}$ is a projection operator onto a one-dimensional manifold of $\mathscr{H}_1 \otimes \cdots \otimes \mathscr{H}_M$;

3. the state vector $|\psi(1,..., N)\rangle$ is factorizable, i.e., there exist a state $|\phi(1,..., M)\rangle$ of $\mathscr{H}_1 \otimes \cdots \otimes \mathscr{H}_M$ and a state $|\xi(M+1,..., N)\rangle$ of $\mathscr{H}_{M+1} \otimes \cdots \otimes \mathscr{H}_N$ such that $|\psi(1,..., N)\rangle = |\phi(1,..., M)\rangle \otimes |\xi(M+1,..., N)\rangle$.

It is worthwhile to generalize Theorem 4.2 to the present situation of N distinguishable particles. We show once again that there is a strict relation between the existence of a projection operator $P^{(1\cdots M)}$ satisfying condition (5.1), i.e., between that fact that the particles fall into two non-entangled sets, and the range $\mathscr{R}[\rho^{(1\cdots M)}]$ of the reduced statistical operator $\rho^{(1\cdots M)}$ of the first M particles of the compound system:

Theorem 5.2. A necessary and sufficient condition for the projection operator $P^{(1\cdots M)}_{\mathscr{M}}$ onto the linear manifold of $\mathscr{H}_1 \otimes \cdots \otimes \mathscr{H}_M$ to satisfy the two following conditions:

1. $Tr^{(1\cdots M)}[P^{(1\cdots M)}_{\mathscr{M}}\rho^{(1\cdots M)}] = 1$;
2. there is no projection operator $\tilde{P}^{(1\cdots M)}_{\mathscr{N}}$ of $\mathscr{H}_1 \otimes \cdots \otimes \mathscr{H}_M$ satisfying the conditions $\tilde{P}^{(1\cdots M)}_{\mathscr{N}} < P^{(1\cdots M)}_{\mathscr{M}}$ (i.e., it projects onto a proper submanifold $\mathscr{N}$ of $\mathscr{M}$) and $Tr^{(1\cdots M)}[\tilde{P}^{(1\cdots M)}_{\mathscr{N}}\rho^{(1\cdots M)}] = 1$,

is that the range $\mathscr{R}[\rho^{(1\cdots M)}]$ of the reduced statistical operator $\rho^{(1\cdots M)}$ coincides with $\mathscr{M}$.

Proof. It is a straightforward generalization of the proof already given for Theorem 4.2. ∎

Leaving aside the case which we have already analyzed in which $\mathscr{M}$ is one-dimensional, once again two possibilities arise:

1. $\mathscr{R}[\rho^{(1\cdots M)}] = \mathscr{M} \subset \mathscr{H}_1 \otimes \cdots \otimes \mathscr{H}_M$,
2. $\mathscr{R}[\rho^{(1\cdots M)}] = \mathscr{H}_1 \otimes \cdots \otimes \mathscr{H}_M$.

While in the first case we can say that the two groups of particles are *partially entangled* since we can attribute to the subsystem $\mathscr{S}_{(1\cdots M)}$ some objective properties (in fact, given an operator $\Omega^{(1\cdots M)}$ which commutes with $P^{(1\cdots M)}_{\mathscr{M}}$ we can consider the spectrum $\mathscr{B}$ of its restriction $\Omega_R = P^{(1\cdots M)}_{\mathscr{M}}\Omega^{(1\cdots M)}P^{(1\cdots M)}_{\mathscr{M}}$ to $\mathscr{M}$, and we can say that subsystem $\mathscr{S}_{(1\cdots M)}$ *has* the objective and (possibly) unsharp properties that Ω_R has a value belonging to $\mathscr{B}$), in the second case subsystem $\mathscr{S}_{(1\cdots M)}$ is *totally entangled* since it does not possess any objective property whatsoever (the only projection operator $P^{(1\cdots M)}$ satisfying condition (5.1) being the identity operator).

5.2. The Case of Completely Non-Entangled Constituents

As already remarked, the constituents of the subsystems $\mathscr{S}_{(1\cdots M)}$ and $\mathscr{S}_{(M+1\cdots N)}$ of the original system may, in turn, be entangled or non-entangled among themselves. One has then to consider the corresponding states $|\phi(1,..., M)\rangle$ and $|\xi(M+1,..., N)\rangle$ and to repeat for them an analysis of the type we have just described. We will not go into details, we will limit ourselves to analyze briefly the case in which all N constituents of the original system are non-entangled with each other.

Definition 5.2. The pure state $|\psi(1,\ldots,N)\rangle \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ is completely non-entangled if there exist N one-dimensional projection operators $P^{(i)}$ belonging to the Hilbert space $\mathcal{H}_i$ respectively, such that:[13]

$$Tr^{(1+\cdots+N)}[P^{(i)}\,|\psi(1,\ldots,N)\rangle\langle\psi(1,\ldots,N)|] = 1 \qquad \forall i = 1\cdots N \tag{5.2}$$

One can then easily prove the following theorems which are straightforward generalizations of those we have proved for the simpler case of two-particle states.

Theorem 5.3. The state $|\psi(1,\ldots,N)\rangle$ is completely non-entangled iff the N reduced density operators $\rho^{(i)} = Tr^{\forall j\neq i}[|\psi(1,\ldots,N)\rangle\langle\psi(1,\ldots,N)|]$, where the trace is calculated with the exclusion of subsystem i, are one dimensional projection operators.

Theorem 5.4. The pure state $|\psi(1,\ldots,N)\rangle \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ is completely non-entangled iff it is completely factorizable, i.e., there exist N states $|\varphi(1)\rangle \in \mathcal{H}_1,\ldots, |\theta(N)\rangle \in \mathcal{H}_N$ such that $|\psi(1,\ldots,N)\rangle = |\varphi(1)\rangle \otimes \cdots \otimes |\theta(N)\rangle$.

5.3. Correlations between the Subsystems

Obviously, one can repeat also in the present case the considerations of Section 4.3 and one can show that in the case in which the first M particles are non-entangled with the remaining $K = N - M$ ones, an equation perfectly analogous to (4.12) holds. In fact

Theorem 5.5. Subsystem $\mathcal{S}_{(1\cdots M)}$ is non-entangled with subsystem $\mathcal{S}_{(M+1\cdots N)}$ iff, given the pure state $|\psi(1\cdots N)\rangle$ of the composite system, the following equation holds for any pair of observables $A(1\cdots M)$ of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_M$ and $B(M+1\cdots N)$ of $\mathcal{H}_{M+1} \otimes \cdots \otimes \mathcal{H}_N$ such that $|\psi(1\cdots N)\rangle$ belongs to their domains:

$$\begin{aligned}&\langle\psi(1\cdots N)|\, A\otimes B\,|\psi(1\cdots N)\rangle\\ &\quad = \langle\psi(1\cdots N)|\, A\,|\psi(1\cdots N)\rangle\langle\psi(1\cdots N)|\, B\,|\psi(1\cdots N)\rangle\end{aligned} \tag{5.3}$$

[13] We observe that it is sufficient to admit that there exist $N-1$ such projection operators, since from the existence of $N-1$ of them it follows that a further operator with the same properties must exist.

From the above equation there follows that the joint probabilities for outcomes of independent measurement processes on the two non-entangled subsystems factorize.

PART III. ENTANGLEMENT AND PROPERTIES OF QUANTUM SYSTEMS OF IDENTICAL PARTICLES IN PURE STATES

This part is devoted to the problem of attributing properties and to the analysis of entanglement of composite systems whose constituents are identical. As we will see such a problem is a quite delicate one and requires a detailed discussion. With the exception of few papers[12–15] this matter has not been adequately discussed in the literature.

6. IDENTITY, INDIVIDUALITY AND PROPERTIES IN QUANTUM THEORY

The so called "principle of individuality" of physical systems has a long history in philosophy, the most naive position about it deriving from the observation that even two extremely similar objects will always display some differences in their properties allowing to distinguish and to individuate the objects. Leibniz has strongly committed himself to such a position by claiming: "*there are never in nature two exactly similar entities in which one cannot find an internal difference*." However, one could try to individuate even absolutely identical objects by taking into account that they differ at least for their location in space and time. In the debate about this problem it is generally agreed that the objects we are interested in should be regarded as individuals. But then a quite natural problem arises: can the fundamental "objects" of current physical theories, such as electrons, protons, etc., be regarded as individuals? And what is the status of such a question within the classical and quantum schemes?

The first obvious remark is that in both schemes such entities are assumed as identical in the sense of possessing precisely all the same intrinsic properties, such as rest mass, charge, spin, magnetic moment, and so on. However, within the framework of classical mechanics each particle follows a perfectly defined trajectory and thus it can (at least in principle) be distinguished from all the others.

The situation is quite different in quantum mechanics, due to the fact that such a theory does not even allow to entertain the idea of particle trajectories and implies that wave functions spread, so that, even if one could label at a certain time one of the two identical particles as 1 and the other as 2, one would not be able, even in principle, to claim, with reference to a subsequent act of detection, whether the particle which has been detected is the one he has labeled 1 or 2.

The above facts have led some philosophers of science to the conclusion that quantum particles cannot be regarded as individuals in any of the traditional meanings of such a term. We will not enter into this relevant question for which we refer the reader to some recent interesting contributions;[(16–19)] we plainly accept, as imposed by the formalism, that when one is dealing with assemblies of identical quantum systems it is simply meaningless to try to "name" them in a way or another.

However, the problem of identifying indiscernible objects is not the relevant one for this paper. What concerns us is the possibility of considering some properties as objectively possessed by quantum systems. Accordingly, it goes without saying that, when dealing with the system, e.g., of two electrons, we will never be interested in questions like "is the electron which we have labeled 1 at a certain position or is its spin aligned with a given axis?", but our only concerns will be of the type: on the basis of the knowledge of the state vector describing the composite system, can one legitimately consider as objective a statement of the kind "there is an electron in a certain region and it has its spin up along a considered direction?" Obviously, in accordance with the position we have taken in Section 3, the above statement must be read as: does the theory guarantee that if a measuring apparatus aimed to reveal an electron and to measure its spin along the considered direction would be activated, it will give with certainty the considered outcomes?

When one takes such a perspective the problem of considering the constituents (we are not interested in which ones) of a system of identical particles as possessing objectively definite properties can be tackled in a mathematically precise way. Correspondingly, one can formulate in a rigorous way the idea that identical particles are non entangled.

We will analyze first of all the case of two identical particles, which represents an ideal arena to point out various subtle aspects of the problem under investigation.

7. ENTANGLEMENT OF TWO IDENTICAL PARTICLES

As just mentioned, in the case of a system containing identical constituents the problem of entanglement has to be reconsidered. In fact, the naive idea that the two systems being non-entangled requires and is guaranteed by the fact that their state vector is the direct product of vectors belonging to the corresponding Hilbert spaces, cannot be simply transposed to the case of interest. One can easily realize that this must be the case by taking into account that the only allowed states for a system of two identical particles must exhibit precise symmetry properties under the exchange of the two particles. If one would adopt the previous criterion one would be led to conclude (mistakenly) that non-entangled states of identical particles cannot exist.[14] The inappropriateness of such a conclusion derives from not taking into account various fundamental facts, in particular that identical particles are truly indistinguishable, so that one cannot pretend that a particular one of them has properties, and that the set of observables for such a system has to be restricted to the self-adjoint operators which are symmetric for the exchange of the variables referring to the two subsystems.

To prepare the ground for such an analysis, we begin by discussing the case of two identical particles which turns out to be simpler. For the moment we will deal simultaneously with the case of identical fermions and bosons. However since some relevant differences occur in the two cases we will subsequently consider them separately in Sections 7.1.1 and 7.1.2.

The necessary modifications of the definitions and theorems of this section when many particles are taken into account will be given at the appropriate stage of our analysis, after a critical reconsideration of the problem of property attribution will be presented.

7.1. Entanglement and Individual Properties of Identical Constituents

As we have noticed in Section 4.1, in the case of distinguishable particles the fact that one of the two constituents of a composite system possesses a complete set of properties automatically implies that the same holds true for the other constituent. While this happens also for identical fermions, it is not true, in general, for systems involving identical bosons. We will strictly link the idea that two identical particles are non entangled to the request that both of them possess a complete set of properties. Accordingly we give the following definition:

Definition 7.1. The identical constituents $\mathscr{S}_1$ and $\mathscr{S}_2$ of a composite quantum system $\mathscr{S} = \mathscr{S}_1 + \mathscr{S}_2$ are **non-entangled** when both constituents possess a complete set of properties.

Taking into account this fact, and in order to deal as far as possible simultaneously with the fermion and the boson cases, we will first of all identify the conditions under which one can legitimately claim that *one of*

[14] We are disregarding here the case of the direct product of two identical state vectors for two identical bosons. Such a case will be discussed in great details in what follows.

the constituents possesses a complete set of properties. Once this will be done, we will separately deal with the problem of the entanglement by distinguishing the fermion from the boson case.

In accordance with the above analysis, let us begin by identifying the necessary and sufficient conditions in order that one of a pair of identical particles possesses a complete set of properties.

Definition 7.2. Given a composite quantum system $\mathscr{S} = \mathscr{S}_1 + \mathscr{S}_2$ of two identical particles described by the pure density operator ρ, we will say that one of the constituents has a complete set of properties iff there exists a one dimensional projection operator P, defined on the Hilbert space $\mathscr{H}^{(1)}$ of each of the subsystems, such that:

$$Tr^{(1+2)}[E(1,2)\,\rho] = 1 \tag{7.1}$$

where

$$E(1,2) = P^{(1)} \otimes I^{(2)} + I^{(1)} \otimes P^{(2)} - P^{(1)} \otimes P^{(2)} \tag{7.2}$$

We stress that the operator $E(1,2)$ is symmetric under the exchange of the labels of the two particles and that it is a projection operator: $[E(1,2)]^2 = E(1,2)$. Furthermore $Tr^{(1+2)}[E(1,2)\,\rho]$ gives the probability of finding *at least* one of the two identical particles in the state onto which the one-dimensional operator P projects, as is immediately checked by noticing that $E(1,2)$ can also be written as[15]

$$E(1,2) = P^{(1)} \otimes [I^{(2)} - P^{(2)}] + [I^{(1)} - P^{(1)}] \otimes P^{(2)} + P^{(1)} \otimes P^{(2)} \tag{7.3}$$

When $E(1,2)$ is multiplied by ρ, the trace of the first term in Eq. (7.3) gives the probability that particle 1 has the property associated to P while the second one does not have such a property, the trace of the second term gives the same probability with particle 1 and 2 interchanged and the trace of the third term gives the probability that both particles have the considered property. Since the above occurrences are mutually exclusive, condition $Tr^{(1+2)}[E(1,2)\,\rho] = 1$ implies that at least one particle has the property under consideration.

[15] We remark that one could drop the last term in the expression (7.3) getting an operator whose expectation value would give the probability of precisely one particle having the properties associated to P. In the case of identical fermions this would make no difference but for bosons it would not cover the case of both particles having precisely the same properties.

It is interesting to relate the fact that one constituent possesses a complete set of properties to the explicit form of the state vector. This is specified by the following theorem:

Theorem 7.1. One of the identical constituents of a composite quantum system $\mathscr{S} = \mathscr{S}_1 + \mathscr{S}_2$, described by the pure normalized state $|\psi(1,2)\rangle$ has a complete set of properties iff $|\psi(1,2)\rangle$ is obtained by symmetrizing or antisymmetrizing a factorized state.

Proof. If $|\psi(1,2)\rangle$ is obtained by symmetrizing or antisymmetrizing a factorized state of two identical particles:

$$|\psi(1,2)\rangle = N[|\varphi^{(1)}\rangle \otimes |\chi^{(2)}\rangle \pm |\chi^{(1)}\rangle \otimes |\varphi^{(2)}\rangle] \tag{7.4}$$

expressing the state $|\chi^{(i)}\rangle$ as follows

$$|\chi^{(i)}\rangle = \alpha\, |\varphi^{(i)}\rangle + \beta\, |\varphi_{\perp}^{(i)}\rangle, \qquad \langle \varphi^{(i)} \,|\, \varphi_{\perp}^{(i)}\rangle = 0 \tag{7.5}$$

and choosing $P = |\varphi\rangle\langle\varphi|$ one gets immediately

$$Tr^{(1+2)}[E(1,2)\,\rho] \equiv \langle\psi(1,2)|\,E(1,2)\,|\psi(1,2)\rangle = \frac{2(1 \pm |\alpha|^2)}{2(1 \pm |\alpha|^2)} = 1 \tag{7.6}$$

Alternatively, since $E(1,2)$ is a projection operator:

$$\begin{aligned}[\langle\psi(1,2)|\,E(1,2)\,|\psi(1,2)\rangle = 1] &\Rightarrow [\|E(1,2)\,|\psi(1,2)\rangle\| = 1] \\ &\Rightarrow [E(1,2)\,|\psi(1,2)\rangle = |\psi(1,2)\rangle]\end{aligned} \tag{7.7}$$

If one chooses a complete orthonormal set of single particle states whose first element $|\Phi_0\rangle$ spans the one-dimensional linear manifold onto which the one dimensional projection operator P projects, writing

$$|\psi(1,2)\rangle = \sum_{ij} c_{ij}\, |\Phi_i^{(1)}\rangle \otimes |\Phi_j^{(2)}\rangle \qquad \sum_{ij} |c_{ij}|^2 = 1 \tag{7.8}$$

and, using the explicit expression for $E(1,2)$ in terms of such a P, one gets:

$$\begin{aligned}E(1,2)\,|\psi(1,2)\rangle = |\Phi_0^{(1)}\rangle \otimes \left[\sum_{j\neq 0} c_{0j}\, |\Phi_j^{(2)}\rangle\right] + \left[\sum_{j\neq 0} c_{j0}\, |\Phi_j^{(1)}\rangle\right] \otimes |\Phi_0^{(2)}\rangle \\ + c_{00}\, |\Phi_0^{(1)}\rangle \otimes |\Phi_0^{(2)}\rangle\end{aligned} \tag{7.9}$$

Imposing condition (7.7), i.e., that the r.h.s. of Eq. (7.9) coincides with $|\psi(1,2)\rangle$ as given by Eq. (7.8), we obtain $c_{ij} = 0$ when both i and j are different from zero. Taking into account that for identical particles $c_{0j} = \pm c_{j0}$, the normalization condition of the state $|\psi(1,2)\rangle$ becomes

$$|c_{00}|^2 + 2 \sum_{j \neq 0} |c_{0j}|^2 = 1 \tag{7.10}$$

We have then shown that:

$$|\psi(1,2)\rangle = |\Phi_0^{(1)}\rangle \otimes \left[\sum_{j\neq 0} c_{0j} \, |\Phi_j^{(2)}\rangle \right] + \left[\sum_{j \neq 0} c_{j0} \, |\Phi_j^{(1)}\rangle \right] \otimes |\Phi_0^{(2)}\rangle$$
$$+ c_{00} \, |\Phi_0^{(1)}\rangle \otimes |\Phi_0^{(2)}\rangle \tag{7.11}$$

In the case of fermions $c_{00} = 0$. Then, introducing a normalized vector $|\Xi^{(i)}\rangle = \sqrt{2} \sum_{j \neq 0} c_{0j} \, |\Phi_j^{(i)}\rangle$ one has

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2}} [|\Phi_0^{(1)}\rangle \, |\Xi^{(2)}\rangle - |\Xi^{(1)}\rangle \, |\Phi_0^{(2)}\rangle] \tag{7.12}$$

with $|\Phi_0^{(k)}\rangle$ and $|\Xi^{(k)}\rangle$ orthogonal.

For bosons, defining the following normalized vector

$$|\Theta^{(i)}\rangle = \sqrt{\frac{4}{2 - |c_{00}|^2}} \left[\sum_{j \neq 0} c_{0j} \, |\Phi_j^{(i)}\rangle + \frac{c_{00}}{2} \, |\Phi_0^{(i)}\rangle \right] \tag{7.13}$$

the two-particle state vector (7.11) becomes

$$|\psi(1,2)\rangle = \sqrt{\frac{2 - |c_{00}|^2}{4}} \, [|\Phi_0^{(1)}\rangle \, |\Theta^{(2)}\rangle + |\Theta^{(1)}\rangle \, |\Phi_0^{(2)}\rangle] \tag{7.14}$$

Note that in this case the states $|\Phi_0^{(k)}\rangle$ and $|\Theta^{(k)}\rangle$ are orthogonal iff the coefficient c_{00} is zero. ▮

There follows that the process of symmetrization or antisymmetrization of a factorized quantum state describing a system composed of identical particles does not forbid to attribute a complete set of physical properties to the subsystems: the only claim that one cannot make is to attribute the possessed property to one rather than to the other constituent.

At this point it is appropriate to deal separately with the case of identical fermions and identical bosons. We will denote the operator $E(1, 2)$ of Eq. (7.2) as $E_f(1, 2)$ and $E_b(1, 2)$ in the two cases, respectively.

7.1.1. The Fermion Case

We analyze first of all the case of two identical fermions, in which, since $P^{(1)} \otimes P^{(2)} = 0$ on the space of totally antisymmetric functions, one can drop such a term in all previous formulae. Accordingly $E_f(1, 2) = P^{(1)} \otimes I^{(2)} + I^{(1)} \otimes P^{(2)}$. Some remarks are appropriate.

As one sees from Eq. (7.12) and in accordance with Definition 7.2, due to the orthogonality of $|\Phi_0\rangle$ and $|\Xi\rangle$, for such a state one can claim not only that there is one fermion possessing the complete set of properties identified by the state $|\Phi_0\rangle$, but also one fermion possessing the complete set of properties identified by the state $|\Xi\rangle$.

According to Definition 7.1, we have thus proved the following theorem for a two fermion system:

Theorem 7.2. The identical fermions $\mathscr{S}_1$ and $\mathscr{S}_2$ of a composite quantum system $\mathscr{S} = \mathscr{S}_1 + \mathscr{S}_2$ described by the pure normalized state $|\psi(1, 2)\rangle$ are non-entangled iff $|\psi(1, 2)\rangle$ is obtained by antisymmetrizing a factorized state.

If, with reference to expression (7.12), we call $P = |\Phi_0\rangle\langle\Phi_0|$ and $Q = |\Xi\rangle\langle\Xi|$ and we define in terms of them the projection operators $E_f(1, 2) = P^{(1)} \otimes I^{(2)} + I^{(1)} \otimes P^{(2)}$ and $F_f(1, 2) = Q^{(1)} \otimes I^{(2)} + I^{(1)} \otimes Q^{(2)}$, we see that

$$\begin{cases} Tr^{(1+2)}[E_f(1, 2)\,|\psi(1, 2)\rangle\langle\psi(1, 2)|] = 1, \\ Tr^{(1+2)}[F_f(1, 2)\,|\psi(1, 2)\rangle\langle\psi(1, 2)|] = 1, \end{cases} \tag{7.15}$$

Moreover, $E_f(1, 2) \cdot F_f(1, 2)$ is a projection operator onto a one-dimensional manifold of the Hilbert space of the two fermions, it coincides with the operator $P^{(1)} \otimes Q^{(2)} + Q^{(1)} \otimes P^{(2)}$ and, as a consequence of the relations (7.15), it satisfies:

$$\begin{aligned} &\langle\psi(1, 2)|\, E_f(1, 2) \cdot F_f(1, 2)\, |\psi(1, 2)\rangle \\ &\qquad \equiv \langle\psi(1, 2)|\, P^{(1)} \otimes Q^{(2)} + Q^{(1)} \otimes P^{(2)}\, |\psi(1, 2)\rangle = 1 \end{aligned} \tag{7.16}$$

Before concluding this section we judge it extremely relevant to call attention to the existence of a certain arbitrariness concerning the properties one can consider as objectively possessed by the constituents of a non entangled state of two fermions. In fact, suppose the state $|\psi(1, 2)\rangle$ has the expression (7.12). Then, if consideration is given to the two dimensional manifold spanned by the single particle states $|\Phi_0\rangle$ and $|\Xi\rangle$, one immediately sees that if one chooses any two other orthogonal single particle states $|\Lambda\rangle$ and

$|\Gamma\rangle$ spanning the same manifold, then $|\psi(1,2)\rangle$ can also be written (up to an overall phase phactor) as:

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2}}\left[|\Lambda^{(1)}\rangle\,|\Gamma^{(2)}\rangle - |\Gamma^{(1)}\rangle\,|\Lambda^{(2)}\rangle\right] \tag{7.17}$$

Obviously, such an expression makes legitimate the assertion that the two fermions have the complete sets of properties associated to $|\Lambda\rangle$ and $|\Gamma\rangle$.

Such a fact might appear as rather puzzling. However as we will see in Section 7.4, it does not give rise to conceptual problems, but it requires to analyze more deeply the situation, to make perfectly clear the subtle interplay between the identity of the constituents and the problem of attributing objective properties to them.

7.1.2. The Boson Case

Let us consider now the boson case. As one sees from Eqs. (7.4) and (7.14), once more the requirement that one of the two identical bosons possesses a complete set of properties implies that the state is obtained by symmetrizing a factorized state. However, there are some remarkable differences with respect to the fermion case. With reference to the expression (7.14) we see that now three cases are possible:

- $|\Theta^{(i)}\rangle \propto |\Phi_0^{(i)}\rangle$. In such a case the state is $|\psi(1,2)\rangle = |\Phi_0^{(1)}\rangle \otimes |\Phi_0^{(2)}\rangle$ and one can claim that "there are two bosons with the complete set of properties associated to $P = |\Phi_0\rangle\langle\Phi_0|$."
- $\langle\Theta^{(i)}|\Phi_0^{(i)}\rangle = 0$, i.e., $c_{00} = 0$. One can then consider the operators $E_b(1,2)$ and $F_b(1,2)$ and their product $E_b(1,2)\,F_b(1,2) = P^{(1)} \otimes Q^{(2)} + Q^{(1)} \otimes P^{(2)}$ where $P = |\Phi_0\rangle\langle\Phi_0|$ and $Q = |\Theta\rangle\langle\Theta|$. Exactly the same argument of the case of two identical fermions makes then clear that one can legitimately claim that "state $|\psi(1,2)\rangle$ represents a system where one of the two bosons has the properties associated to the projection operator P *and* one those associated to the projection operator Q."
- Finally it can happen that $\langle\Theta^{(i)}\,|\,\Phi_0^{(i)}\rangle \neq 0$ but $|\Theta^{(i)}\rangle$ is not proportional to $|\Phi_0^{(i)}\rangle$. Then, even though we can state that "there is a boson with the properties associated to the projection operator P" as well as "there is a boson with the properties associated to the projection operator Q," we cannot state that "the state (7.14) describes a system in which one of the bosons has the properties associated to P *and* the other those associated to Q." Actually in the considered case there is a non vanishing probability of finding both particles in the same state.

According to our Definition 7.1 which, as already stated, we adopt completely in general for systems of identical particles, we see that in the last of the just considered cases we cannot assert that the two bosons are non entangled, while we can do so for the first two cases. The following theorem has thus been proved:

Theorem 7.3. The identical bosons of a composite quantum system $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$ described by the pure normalized state $|\psi(1,2)\rangle$ are **non-entangled** iff either the state is obtained by symmetrizing a factorized product of two orthogonal states or if it is the product of the same state for the two particles.

Before concluding, we point out that in the boson case and when $|\psi(1,2)\rangle$ is obtained by symmetrizing two orthogonal vectors, contrary to what happens for two fermions, the two states are perfectly defined (up to a phase factor), i.e., there are no other orthogonal states $|\alpha\rangle$ and $|\beta\rangle$ differing from $|\Phi_0\rangle$ and $|\Theta\rangle$, such that one can write $|\psi(1,2)\rangle$ in the form:

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2}}[|\alpha^{(1)}\rangle\,|\beta^{(2)}\rangle + |\beta^{(1)}\rangle\,|\alpha^{(2)}\rangle] \tag{7.18}$$

The proof is easily derived along the following lines. Writing[16]

$$\begin{cases} |\alpha^{(i)}\rangle = a\,|\Phi_0^{(i)}\rangle + b\,|\Theta^{(i)}\rangle \\ |\beta^{(i)}\rangle = -b^*\,|\Phi_0^{(i)}\rangle + a^*\,|\Theta^{(i)}\rangle \end{cases} \tag{7.19}$$

with $|a|^2 + |b|^2 = 1$, we get from Eq. (7.18):

$$\begin{aligned} |\psi(1,2)\rangle = \frac{1}{\sqrt{2}}[&-2ab^*\,|\Phi_0^{(1)}\rangle\,|\Phi_0^{(2)}\rangle + (|a|^2 - |b|^2)(|\Phi_0^{(1)}\rangle\,|\Theta^{(2)}\rangle \\ &+ |\Theta^{(1)}\rangle\,|\Phi_0^{(2)}\rangle) + 2ba^*\,|\Theta^{(1)}\rangle\,|\Theta^{(2)}\rangle] \end{aligned} \tag{7.20}$$

which can coincide with

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2}}[|\Phi_0^{(1)}\rangle\,|\Theta^{(2)}\rangle + |\Theta^{(1)}\rangle\,|\Phi_0^{(2)}\rangle] \tag{7.21}$$

iff $a = 0$ or $b = 0$, implying that $|\alpha\rangle$ is $|\Phi_0\rangle$ and $|\beta\rangle$ is $|\Theta\rangle$ or viceversa. ∎

7.1.3. Concluding Remarks

The conclusion of this section is that the concept of entanglement can be easily generalized to the case of two identical quantum subsystems provided one relates it to the possibility of attributing complete sets of

[16] Note that given $|\alpha^{(i)}\rangle$ we have chosen a precise phase factor in defining $|\beta^{(i)}\rangle$

objective properties to both constituents. The theorems of Section 7.1 make absolutely precise the mathematical aspects characterizing the states which can be considered as describing non-entangled systems, i.e., the fact that: (i) the states for the whole system must be obtained by appropriately (anti)-symmetrizing a factorized state of the two particles; (ii) the factors of such states must be orthogonal in the fermion case and they can be either orthogonal or equal in the boson case.

Obviously, the above conclusion implies that entangled states of two identical particles can very well occur. Just to give an example we can consider the following state of two spin-1/2 particles:

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2}}[|\vec{n}\uparrow\rangle_1\,|\vec{n}\downarrow\rangle_2 - |\vec{n}\downarrow\rangle_1\,|\vec{n}\uparrow\rangle_2] \otimes |\omega(1,2)\rangle \tag{7.22}$$

$|\omega(1,2)\rangle$ being a symmetric state of $\mathscr{L}(R^3) \otimes \mathscr{L}(R^3)$. State (7.22) cannot be written as a symmetrized product of two orthogonal states, and, consequently no constituent possesses any conceivable complete set of (internal and spatial) properties.

7.2. Sharp and Unsharp Properties

In our discussion concerning the properties of one of a pair of identical constituents we have focussed our attention on complete set of properties. The formalization of this idea consists in assuming that there exists a single particle projection operator P onto a one-dimensional manifold such that the projection operator of Eq. (7.2) satisfies condition (7.1). Obviously we could have played an analogous game by considering a projection operator $P_{\mathscr{M}}$ of the single particle Hilbert space $\mathscr{H}^{(1)}$ projecting onto a multidimensional submanifold $\mathscr{M}$ of such a space. Suppose that, with this choice, the corresponding operator:

$$E_{\mathscr{M}}(1,2) = P^{(1)}_{\mathscr{M}} \otimes (I^{(2)} - P^{(2)}_{\mathscr{M}}) + (I^{(1)} - P^{(1)}_{\mathscr{M}}) \otimes P^{(2)}_{\mathscr{M}} + P^{(1)}_{\mathscr{M}} \otimes P^{(2)}_{\mathscr{M}} \tag{7.23}$$

satisfies:

$$Tr[E_{\mathscr{M}}(1,2)\,|\psi(1,2)\rangle\langle\psi(1,2)|] = 1 \tag{7.24}$$

As it is immediate to see one can consider an arbitrary single particle self-adjoint operator Ω commuting with $P_{\mathscr{M}}$, and consider the restriction $\omega(\mathscr{M})$ of its spectrum to the manifold $\mathscr{M}$. Then the validity of (7.24) implies that we can state that one of the particles has the property that the value of Ω belongs to $\omega(\mathscr{M})$.

Thus, in the considered case, even though we cannot attribute to any of the particles a complete set of properties, we can, in general, attribute to it unsharp properties. If the only manifold $\mathscr{M}$ for which the above situation holds is the whole Hilbert space, we can state that the particles possess no sharp or unsharp properties at all.

In complete analogy to what has been said in Section 4.2 concerning the various degrees of entanglement occurring in systems of distinguishable particles, it is worthwhile to summarize the situation in the following way, where $|\psi(1,2)\rangle$ represents an arbitrary state of two identical particles:

- there exists a one dimensional projection operator P of $\mathscr{H}^{(1)}$ such that $Tr[E(1,2)\,|\psi(1,2)\rangle\langle\psi(1,2)|] = 1 \Rightarrow$ one subsystem possesses a complete set of properties;

- there exists a projection operator $P_{\mathscr{M}}$ projecting onto a proper submanifold $\mathscr{M} \subset \mathscr{H}$ of dimension greater than 1, such that $Tr[E_{\mathscr{M}}(1,2)\,|\psi(1,2)\rangle\langle\psi(1,2)|] = 1 \Rightarrow$ one subsystem possesses some properties, but not a complete set of them;

- there exists no projection operator $P_{\mathscr{M}}$ projecting onto a proper submanifold of $\mathscr{H}^{(1)}$ such that $Tr[E_{\mathscr{M}}(1,2)\,|\psi(1,2)\rangle\langle\psi(1,2)|] = 1 \Rightarrow$ one subsystem (actually both of them) does not possess any property at all.

7.3. Correlations in the Case of Two Identical Particles

In this subsection we reconsider briefly the problem of the correlations of two particles in entangled or non-entangled states in the case in which they are identical. Before coming to a detailed analysis let us stress that the problem under consideration has a particular relevance in the specific case in which the two particles are in different spatial regions, since this is the case in which the problem of the nonlocal aspects of the formalism emerges as a central one. Let us then consider two identical particles with space and internal degrees of freedom and let us denote as $\mathscr{H}_{\text{sp}}(1,2)$ and $\mathscr{H}_{\text{int}}(1,2)$ the corresponding Hilbert spaces. The Hilbert space for the whole system is, obviously, the appropriate symmetric or antisymmetric submanifold $\mathscr{H}_{S,A}(1,2)$ of the space $\mathscr{H}_{\text{sp}}(1,2) \otimes \mathscr{H}_{\text{int}}(1,2)$. Let us also assume that the pure state associated to the composite system is obtained by (anti)symmetrizing a factorized state of the two particles corresponding to their having different spatial locations. To be explicit, we start from a state:

$$|\psi_{\text{fact}}(1,2)\rangle = |\varsigma\rangle_1\,|R\rangle_1\,|\chi\rangle_2\,|L\rangle_2 \tag{7.25}$$

where $|\varsigma\rangle$ and $|\chi\rangle$ are two arbitrary states of the internal space of a particle and $|R\rangle$ and $|L\rangle$ are two orthogonal states whose spatial supports are compact, disjoint and far away from each other. This situation is the one of interest for all experiments about the non-local features of quantum states. From the state (7.25) we pass now to the properly (anti)symmetrized one:

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2}}[|\varsigma\rangle_1\,|R\rangle_1\,|\chi\rangle_2\,|L\rangle_2 \pm |\chi\rangle_1\,|L\rangle_1|\varsigma\rangle_2\,|R\rangle_2] \tag{7.26}$$

Note that we already know that if consideration is given to the operators

$$\begin{aligned} E(1,2) &= P^{(1)} \otimes I^{(2)} + I^{(1)} \otimes P^{(2)} - P^{(1)} \otimes P^{(2)}, \qquad & P &= |\varsigma\rangle\,|R\rangle\langle R|\,\langle\varsigma| \\ F(1,2) &= Q^{(1)} \otimes I^{(2)} + I^{(1)} \otimes Q^{(2)} - Q^{(1)} \otimes Q^{(2)}, \qquad & Q &= |\chi\rangle\,|L\rangle\langle L|\,\langle\chi| \end{aligned} \tag{7.27}$$

the following equations hold:

$$\begin{aligned} Tr^{(1+2)}[E(1,2)\,|\psi(1,2)\rangle\langle\psi(1,2)|] &= 1, \\ Tr^{(1+2)}[F(1,2)\,|\psi(1,2)\rangle\langle\psi(1,2)|] &= 1 \end{aligned} \tag{7.28}$$

which guarantee that the properties related to the projection operators P and Q can be considered as objectively possessed. However, here we are interested in what the theory tells us concerning the correlations between the outcomes of measurement processes on the constituents. To this purpose, we consider two arbitrary observables $\Omega^{(1)}$ and $\Sigma^{(2)}$ of the internal space of the particles and we evaluate the expectation value:

$$\begin{aligned} &\langle\psi|\,[\Omega^{(1)}\,|R\rangle_{1\,1}\langle R| \otimes I^{(2)} + I^{(1)} \otimes \Omega^{(2)}\,|R\rangle_{2\,2}\langle R|] \\ &\quad \times [\Sigma^{(1)}\,|L\rangle_{1\,1}\langle L| \otimes I^{(2)} + I^{(1)} \otimes \Sigma^{(2)}\,|L\rangle_{2\,2}\langle L|]\,|\psi\rangle = \langle\varsigma|\,\Omega\,|\varsigma\rangle\langle\chi|\,\Sigma\,|\chi\rangle \end{aligned} \tag{7.29}$$

Equation (7.29) shows that the "properties referring to the internal degrees of freedom" factorize, just as in the case of two distinguishable particles. Obviously, the same conclusion does not hold when the state is not of the considered type, e.g., when it is a genuinely entangled state such as:

$$|\psi(1,2)\rangle = \tfrac{1}{2}[|\varsigma\rangle_1\,|\chi\rangle_2 - |\chi\rangle_1\,|\varsigma\rangle_2] \otimes [|R\rangle_1\,|L\rangle_2 \pm |L\rangle_1\,|R\rangle_2] \tag{7.30}$$

The conclusion should be obvious: also from the point of view of the correlations, and consequently of the implications concerning nonlocality, the non-entangled states of two identical particles have the same nice features of those of two distinguishable particles.

7.4. Deepening the Investigation

The analysis of the previous subsections has clarified the situation concerning systems of two identical particles, making precise which is the appropriate way to pose the problem of attributing objective properties and of the entanglement within such a context. The relations between the physical and the formal aspects of such a problem have also been discussed. However, some delicate questions which deserve a further analysis have been naturally raised. This subsection is devoted to deal which such matters. We will begin by trying to make clear, resorting to elementary physical examples, some subtle points which could give rise to misunderstandings. Subsequently we will reconsider the problems we have already mentioned, arising from the arbitrariness about the properties which can be considered as possessed in the case of identical fermions.

7.4.1. Clarifying the Role of the Spatial and Internal Degrees of Freedom

Let us consider a system of two identical spin $1/2$ particles. We stress that if one would confine his attention to the spin degrees of freedom alone, then, following our definitions and theorems, one would be led to conclude that the singlet state,which can be obtained by antisymmetrizing, e.g., the state $|z\uparrow\rangle_1\,|z\downarrow\rangle_2$, would be a non entangled state. How does this fit with our previous remarks and the general (and correct) position that such a state is, in a sense, the paradigmatic case of an entangled two body system? We have already called attention to the necessity of taking also into account, e.g., the position of the constituents, to legitimately raise the relevant questions about their properties. But the matter must be analyzed on more general grounds. In analogy with state (7.25) of the previous section we consider a factorized state of the type

$$|\psi(1,2)\rangle = |z\uparrow\rangle_1\,|R_i\rangle_1\,|z\downarrow\rangle_2\,|R_j\rangle_2 \tag{7.31}$$

with $\langle R_i \mid R_j\rangle = \delta_{ij}$. Now we can make our point: even though it is meaningless (within a quantum context) to speak of particle 1 as distinguishable from particle 2, we can “individuate” the identical objects by resorting to the different spatial quantum numbers i and j. Concerning the state obtained from (7.31) by the antisymmetrization procedure we are sure that one particle (we do not know which one) has the spatial property associated to the quantum number i and one has the property associated to the quantum number j. It is then meaningful to raise the question of the relations existing between the internal properties and the spatial properties. In the considered case we can use each of the differing quantum numbers to “individuate” the constituents and raise, e.g., the question: has the particle identified by the quantum number i, definite spin properties? The answer is obviously affirmative; in our case it definitely has spin up along the z-axis. Note that we could also have used the spin quantum numbers to “individuate” the particles and we could have raised the question: does the particle with spin up along z have precise spatial properties? And the answer would have been yes: it has the spatial properties associated to the state $|R_i\rangle$.

On the contrary, for a state like

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2}}\left[|z\uparrow\rangle_1\,|z\downarrow\rangle_2 - |z\downarrow\rangle_1\,|z\uparrow\rangle_2\right] \otimes \left[|R_i\rangle_1|R_j\rangle_2 + |R_j\rangle_1\,|R_i\rangle_2\right] \tag{7.32}$$

which is not obtainable by antisymmetrizing a factorized state, it is not possible, for example, to attribute any definite spin property to the particle identified by the index i and equivalently no definite spatial property can be attributed to the particle with spin up. In the case where $|R_i\rangle$ and $|R_j\rangle$ correspond to two distant spatial location, the vector (7.32) represents the paradigmatic state considered in the usual EPR argument and in the experiments devised to reveal the non-local features of quantum mechanics.

The picture should now be clear: no state of two fermions in the singlet spin state can be obtained by antisymmetrizing a factorized wave function, when also the remaining degrees of freedom are taken into account. In this sense, and paying the due attention to the subtle problems we have discussed, one can understand how there is no contradiction between the usual statement that the singlet state is entangled and the fact that, if one disregards the spatial degrees of freedom, it can be obtained by antsymmetrizing a factorized spin state.

7.4.2. More About the Case of Two Identical Particles

In Section 7.1 we have shown that, in the case of identical particles, property attribution is legitimate iff the state is obtained by symmetrizing or antisymmetrizing an appropriate factorized state. However, in the fermion case the request that the state can be written in the form

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2}}\left[|\Lambda^{(1)}\rangle\,|\Gamma^{(2)}\rangle - |\Gamma^{(1)}\rangle\,|\Lambda^{(2)}\rangle\right] \tag{7.33}$$

where $|\Lambda\rangle$ and $|\Gamma\rangle$ are two arbitrary orthogonal vectors of the single particle Hilbert space $\mathscr{H}^{(1)}$ leaves some indefiniteness concerning the possessed properties and compels us to face the problem arising from this arbitrariness.

In fact, on the one hand, according to the position we have taken in this paper, which is perfectly in line with the one of Einstein, i.e., that "when one can predict the outcome of a prospective measurement with certainty, there is an element of physical reality associated to it," all claims of the type "one fermion possesses the complete set of properties associated to $|\Gamma\rangle$ and the other those associated to $|\Lambda\rangle$" are perfectly legitimate for the state (7.33). On the other hand this might appear, at first sight, quite embarrassing when one takes into account that the properties we are considering, when we change the states in terms of which we express the unique state $|\psi(1,2)\rangle$ of the composite system, may be very well incompatible among themselves, in the quantum mechanical sense.

However, there are at least two reasons for which one can ignore this, at first sight, puzzling situation, one of formal and physical nature, the second having more to do with the laboratory practice. The general reason derives from the fact that within quantum mechanics it may very well happen that incompatible observables have common eigenstates. For instance, with precise reference to the case under discussion, if consideration is given to the infinitely many noncommuting number operators $N_\lambda = a_\lambda^\dagger a_\lambda$ counting the number of fermions in an arbitrary single particle state $|\lambda\rangle$ of the two dimensional manifold spanned by $|\Lambda\rangle$ and $|\Gamma\rangle$, the state $|\psi(1,2)\rangle$ is a simultaneous eigenstate of all the N_λ's belonging to the eigenvalue 1. This implies, according to the quantum mechanical rules that any apparatus devised to measure whether there are fermions in such a state, will give with certainty the outcome 1, i.e., it will allow to conclude that "there is one fermion in such a state." No matter how peculiar this situation might appear, it is a clear cut consequence of the formalism and of the criterion for attributing properties to physical systems.[17]

Coming now to the relevant practical aspects of the problem we stress once more that, for what concerns entanglement and property attribution, the physically most significant and interesting aspect is the one of the non-local correlations between distant and noninteracting particles occurring in connection with entangled systems. To illustrate this point we can make reference to the following state of two identical fermions:

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2}}\left[|z\uparrow\rangle_1\,|R\rangle_1\,|z\downarrow\rangle_2\,|L\rangle_2 - |z\downarrow\rangle_1\,|L\rangle_1\,|z\uparrow\rangle_2\,|R\rangle_2\right] \qquad (7.34)$$

[17] We consider it appropriate to call attention to a fact that makes the situation even less embarrassing than it might appear. To this purpose, let us consider two arbitrary non-orthogonal vectors $|\lambda\rangle$ and $|\gamma\rangle$ of our two dimensional manifold and the associated projection operators P_λ and P_γ. In terms of them we build the projection operators $E_\lambda(1,2)$ and $E_\gamma(1,2)$, with obvious meaning of the symbols. Suppose now we perform a measurement aimed to ascertain whether there is one fermion in state $|\lambda\rangle$, i.e., we measure $E_\lambda(1,2)$, or, equivalently, we measure the observable N_λ. We then get for sure the eigenvalue 1 and, and this is the crucial point, the measurement does not alter in any way the state vector. This means that the probability of finding, in a subsequent measurement, one fermion in the state $|\gamma\rangle$ is still equal to one, and has not been influenced by the first measurement.

For such a state, as we have already discussed in Section 7.4.1, it is perfectly legitimate, if we attach a prominent role to positions, to claim that "there is a particle at R and it has spin up along the z-axis" as well as that "there is a particle at L and it has spin down along the z-axis." The possibility of making such a claim is the characteristic feature which makes such a state basically different from the state (7.22) or from the singlet state of the EPR set-up. One cannot however avoid recognizing that, if some meaning would be attached to single particle states like:

$$|\Gamma\rangle = \frac{1}{\sqrt{2}}[|z\uparrow\rangle\,|R\rangle + |z\downarrow\rangle\,|L\rangle], \qquad |\Lambda\rangle = \frac{1}{\sqrt{2}}[|z\uparrow\rangle\,|R\rangle - |z\downarrow\rangle\,|L\rangle] \tag{7.35}$$

which do not correspond either to definite locations or to definite spin properties of a particle, then one could claim that in state (7.34) "there is a particle with the properties associated to $|\Gamma\rangle$ *and* one with the properties associated to $|\Lambda\rangle$." It goes without saying that measurements involving states like those of Eq. (7.35) are extremely difficult to perform and of no practical interest.

7.5. Concerning some Misconceptions about Entanglement for Systems of Identical Particles

In the literature one can find some inappropriate statements about entanglement in the case of systems whose constituents are identical. Such misconceptions derive from not having appropriately taken into account the real physical meaning and implications of entanglement. In its essence, the characteristic trait of entanglement derives from the fact that a system which is composed of two subsystems is associated to a state vector such that the subsystems have only "potentialities" concerning most or even all conceivable observables, potentialities which are immediately actualized when one performs a measurement on one of the two subsystems (the most striking situation being connected to instantaneous actualization at-a-distance). For distinguishable particles, as we have seen, such an occurrence is strictly related to the fact that the state vector be nonfactorized.

It is therefore not surprising that some authors have been inappropriately led to identify entanglement with factorizability. However, suppose that in the case of two distinguishable particles, one starts from a state like (7.31) which is manifestly factorized and, consequently, makes legitimate to state that particle 1 has spin up along the z-axis and is in the eigenstate $|R_i\rangle$ of an appropriate observable $\Omega^{(1)}$ pertaining to the eigenvalue ω_i, while particle 2 has spin down along the z-axis and is in the eigenstate $|R_j\rangle$ of the observable $\Omega^{(2)}$ pertaining to the eigenvalue ω_j.

Suppose now that the two particles are identical fermions, so that one properly antisymmetrizes the above state. Then, the resulting state vector is formally no longer factorized, but it is non-entangled since, as we stress once more taking the risk of being pedantic, it makes perfectly legitimate to make the joint statement that "there is one fermion with spin up and the property $\Omega = \omega_i$ and one fermion with spin down and the property $\Omega = \omega_j$." Moreover, the act of measuring one of the two "properties" does not change in any way the fact that the other property can be considered as objectively possessed both before and after the measurement process. This is the reason for which the state *must be claimed* to be non-entangled.

We call attention to the fact that the state being non-entangled is an intrinsic property characterizing it, as follows from the analysis of the previous sections, and does not depend in any way on the basis or the formal apparatus we choose to describe it.

It is just due to a failure of fully appreciating the above facts that one can be led to make misleading statements. As an example, in ref. 20 it is stated that "*one may not draw conclusions about entanglement in configuration space by looking at the states in Fock space*." This statement is based on the fact that, according to the authors, in the case of two bosons, the state:

$$|\phi\rangle = |1_{\vec{k}}\ 1_{\vec{l}}\rangle \tag{7.36}$$

which describes two identical particles with momenta $\vec{k}$ and $\vec{l}$, is a factorizable state in Fock space, being instead an entangled one if one looks at its form in terms of the momentum basis for single particles. This argument is clearly in contradiction with what we have just pointed out. In fact, the state $|\phi\rangle$, if we indicate with $a^{\dagger}(\vec{k})$ the creation operator of a boson with momentum $\vec{k}$, has the following Fock representation and, equivalently, the following expression in terms of a single particle momentum basis:

$$|\phi\rangle = |1_{\vec{k}}\ 1_{\vec{l}}\rangle \equiv \frac{1}{\sqrt{2}} a^{\dagger}(\vec{k})\, a^{\dagger}(\vec{l})\, |0\rangle \Leftrightarrow \frac{1}{\sqrt{2}} [|\vec{k}\rangle_1\, |\vec{l}\rangle_2 + |\vec{l}\rangle_1\, |\vec{k}\rangle_2] \tag{7.37}$$

As the formula shows, the state is obtained by a process of symmetrization of a factorized state of two "orthogonal" states, and as such it is non-entangled. In particular it is perfectly legitimate to claim that there exists with certainty a boson with momentum $\vec{k}$ and one with momentum $\vec{l}$.

This means that states which are "factorized" in Fock space have precisely the same physical properties as those following from their explicit form in configuration or momentum space. Being non-entangled, in the case of two identical particles is, just as in the case of two distinguishable particles, a property which has nothing to do with the way one chooses to express the state vector.

8. ENTANGLEMENT OF *N* INDISTINGUISHABLE PARTICLES

We analyze here the case of N indistinguishable particles. In particular we will be concerned with the analog of the question we have discussed in the case of N distinguishable particles: can one single out a "subgroup" of the constituents (obviously we cannot identify them) to which one can attach a complete set of properties as objectively possessed? The problem is conceptually a rather delicate one and requires remarkable care. Moreover, it has to be stressed that it has a great conceptual and practical relevance. For instance we can be naturally led to face a situation like the following: there is a Helium atom here and a Lithium atom there (in a distant region). We then must pretend that a claim of the kind "this one is a Helium atom" (or, as we will see, in general, one coinciding with it to an extreme—and controllable—degree of accuracy) can be made consistently, in spite of the fact that the correct wave function is totally antisymmetric under the exchange of the electrons of the Helium and Lithium atoms.

Besides these physical aspects we are mainly interested in defining in a conceptually correct way the idea that the set of N identical particles we are dealing with can be partitioned into two "subsets" of cardinality M and K, which are non-entangled with each other.[18] By following strictly the procedure we have introduced for the case of two particles we will do this by considering the possibility of attributing a complete set of properties to each subset and we will give the following definition:

[18] The authors of ref. 18 have remarked that, in spite of their indistinguishability, "*the electrons of an atom, taken as a whole, possess some properties which are characteristic of a set. For instance, they have a cardinality, even if we cannot count them, hence we cannot make an ordinal number to correspond to each electron.*" For this reason they have appropriately introduced the terminology quaset (abbreviation for quasi-set) for a collection of quantum elements which are indistinguishable from each other. The authors have also called attention to the possibility of considering subquasets, by identifying their elements on the basis of their sharing a specific single-particle property. As a typical example they consider the electrons in the shell $2p$ of an atom as a subquaset. In the analysis we are performing, we will deal with a strictly analogous situation, and so, to be rigorous, we should speak of quasets and subquasets. Moreover, for a satisfactory description of the situation we are going to tackle, we should enlarge the idea of subquasets by making reference to a "subgroup of particles" which are related only as a whole to a precise property. In a sense, we will not make reference to the individual elements which have precise individual properties, but to the subquaset which has a global property. Our generalization is, at any rate, strictly related to the one of the authors of ref. 18, since, as we will see, when we will split a quaset, e.g., of N fermions into two subquasets of cardinality M and K, respectively, we will be lead to assume that there exists an appropriate single particle basis such that the two quasets involve two disjoint subsets of the elements of this single particle basis. In the paper, for simplicity, we will not use systematically the appropriate terminology of quasets, and we will speak, quite loosely, of "groups of particles." We believe the reader will have clear what we have in mind, allowing us to avoid resorting to the use of a terminology which is yet not usual in the analysis of entanglement.

Definition 8.1. Given a quantum system of N identical particles described by a pure state $|\psi^{(N)}\rangle$ we will say that it contains two **non-entangled** "subgroups" of particles of cardinality M and K ($M+K=N$), when both subgroups possess a complete set of properties.

The conditions under which it will be possible to attribute a complete set of properties to a quantum subsystem will be made mathematically precise in the following subsections.

As we will see, to make statements of the sort we are interested in, i.e., that objective properties can be attached to the subsets associated to a partition of the particles or, equivalently, that such subsets are non-entangled among themselves, we have to impose quite strict constraints on the state vector of the whole system. After having identified them in a very precise manner, we will be able to evaluate how well they are satisfied in practice and, correspondingly, we will be in the position of judging the degree of legitimacy of our claims concerning precise physical situations.

Here we will deal, from the very beginning, separately with the fermion and boson cases.

9. IDENTICAL FERMIONS

To analyze this problem it is appropriate to begin by fixing our notation and by deriving some simple results which we shall need in what follows. We will deal with a system of N identical fermions and with subsystems of such a system.

9.1. Some Mathematical Preliminaries

We denote as $\mathscr{H}_A^{(R)}$ the Hilbert space which is appropriate for a system of R identical fermions, i.e., the space of the totally skew-symmetric states $|\psi(1,\ldots,R)\rangle$ of the variables (e.g spatial and internal) of the constituents. Obviously $\mathscr{H}^{(1)}$ is the space of single particle states. Let us denote as $\{|\varphi_i\rangle\}$ a complete orthonormal set in such a space. A basis for $\mathscr{H}_A^{(N)}$ is then obtained by antisymmetrizing and normalizing the product states $\{|\varphi_{i_1}(1)\rangle \otimes \cdots \otimes |\varphi_{i_N}(N)\rangle\}$ which, when the subscripts take all the allowed values, are a basis of $\mathscr{H}^{(1)} \otimes \cdots \otimes \mathscr{H}^{(1)}$. For simplicity let us introduce, as usual, the linear antisymmetrization operator A which acts in the following way on the states $\{|\varphi_{i_1}(1)\rangle \otimes \cdots \otimes |\varphi_{i_N}(N)\rangle\}$:

$$A\{|\varphi_{i_1}(1)\rangle \cdots |\varphi_{i_N}(N)\rangle\} \equiv \sum_P (-)^p\, P\{|\varphi_{i_1}(1)\rangle \cdots |\varphi_{i_N}(N)\rangle\} \tag{9.1}$$

where the sum is extended to all permutations P of the variables $(1,..., N)$ —or equivalently of the subscripts $(i_1,..., i_N)$—and p is the parity of the permutation P. As it is well known $A\{|\varphi_{i_1}(1)\rangle \cdots |\varphi_{i_N}(N)\rangle\}$ can be simply expressed as the determinant of an appropriate matrix. The states (9.1) are not normalized, their norm being equal to $\sqrt{N!}$, so that the basis generated in the above way is given by the states $\frac{1}{\sqrt{N!}} A\{|\varphi_{i_1}(1)\rangle \cdots |\varphi_{i_N}(N)\rangle\}$.

We will not use directly such states to express the most general state of $\mathscr{H}_A^{(N)}$, but we will write it as

$$|\psi(1,..., N)\rangle = \sum_{i_1,..., i_N} a_{i_1 \cdots i_N} |\varphi_{i_1}(1)\rangle \cdots |\varphi_{i_N}(N)\rangle \tag{9.2}$$

where the coefficients $a_{i_1 \cdots i_N}$ are totally skew-symmetric and are chosen in such a way that $|\psi(1,..., N)\rangle$ turns out to be normalized, i.e., they satisfy:

$$a_{P(i_1 \cdots i_N)} = (-)^p a_{i_1 \cdots i_N}; \qquad \sum_{i_1,..., i_N} |a_{i_1 \cdots i_N}|^2 = 1 \tag{9.3}$$

In the first of the above relations P represents an arbitrary permutation of the subscripts of $a_{i_1 \cdots i_N}$, and p the parity of the considered permutation.

From now on we will deal with $\mathscr{H}_A^{(N)}$ and we will be interested in "splitting" the N identical constituents into two "subsets" (with reference to their cardinality) of M and $K = N - M$ particles. We begin by recalling a trivial fact, i.e, that the Hilbert space $\mathscr{H}_A^{(N)}$ is a closed linear submanifold of the direct product $\mathscr{H}_A^{(M)} \otimes \mathscr{H}_A^{(K)}$. This follows trivially from Laplace's formula for determinants which can be written as:

$$\begin{aligned} &A\{|\varphi_{i_1}(1)\rangle \cdots |\varphi_{i_M}(M)\rangle\, |\varphi_{r_1}(M+1)\rangle \cdots |\varphi_{r_K}(N)\rangle\} \\ &\quad = \mathscr{G}[A\{|\varphi_{i_1}(1)\rangle \cdots |\varphi_{i_M}(M)\rangle\}\, A\{|\varphi_{r_1}(M+1)\rangle \cdots |\varphi_{r_K}(N)\rangle\}] \end{aligned} \tag{9.4}$$

where the symbol $\mathscr{G}$ at the r.h.s. indicates that one has to sum over all the permutations between the first M particles and the remaining ones, attaching to the various terms the appropriate sign. The above formula shows that the elements of a basis of $\mathscr{H}_A^{(N)}$ can be expressed in terms of the direct products of the elements of two orthonormal complete sets of $\mathscr{H}_A^{(M)}$ and $\mathscr{H}_A^{(K)}$. Since, when consideration is given to two states which have common single-particle indices the antisymmetrization procedure yields the zero vector $|\omega\rangle$ of $\mathscr{H}_A^{(N)}$, the claim that $\mathscr{H}_A^{(N)} \subset \mathscr{H}_A^{(M)} \otimes \mathscr{H}_A^{(K)}$ follows.

9.1.1. Defining an Appropriate Single Particle Basis with Reference to a Given State of M Fermions

Given our system of N identical fermions, we pick up M of them, and we consider a state $|\Pi^{(M)}(1\cdots M)\rangle \in \mathscr{H}_A^{(M)}$ whose Fourier decomposition on the product basis of the single particle states $\{|\varphi_i\rangle\}$ is:

$$|\Pi^{(M)}(1\cdots M)\rangle = \sum_{i_1\cdots i_M} a_{i_1\cdots i_M}\, |\varphi_{i_1}(1)\rangle\cdots|\varphi_{i_M}(M)\rangle \tag{9.5}$$

We then choose an arbitrary normalized single particle state $|\Phi^{(1)}\rangle$, we represent it on the chosen single particle basis,

$$|\Phi^{(1)}\rangle = \sum_t b_t\, |\varphi_t\rangle \tag{9.6}$$

and, with reference to the state (9.5), we define the following subset $V_\perp^{\Pi 1}$ of $\mathscr{H}^{(1)}$:

$$V_\perp^{\Pi 1} \equiv \left\{ |\Phi^{(1)}\rangle \,\middle|\, \sum_t b_t^* a_{ti_2\cdots i_M} = 0,\ \forall i_2,\ldots, i_M \right\} \tag{9.7}$$

We note that $V_\perp^{\Pi 1}$ is independent from the single particle basis we have used to identify it and from the index which is saturated in Eq. (9.7).

The reader will have no difficulty in realizing that $V_\perp^{\Pi 1}$ is a closed linear submanifold[19] of $\mathscr{H}^{(1)}$. It is useful to mention that another way to characterize $V_\perp^{\Pi 1}$ is the following. Suppose we use the shorthand notation $\int dX$ to denote an integral over the space and a summation over the internal variables of the fermion X. Then Eq. (9.7) can be written as

$$V_\perp^{\Pi 1} \equiv \left\{ |\Phi^{(1)}\rangle \,\middle|\, \int d1 \Phi^{(1)*}(1)\, \Pi^{(M)}(1, 2,\ldots, M) = 0 \right\} \tag{9.8}$$

[19] It must be noted that it may very well happen that such a manifold turns out to contain only the zero vector of $\mathscr{H}^{(1)}$. To see this we consider for simplicity the case $M = 2$ and we write $|\Pi^{(2)}(1, 2)\rangle$ as in Eq. (9.5), $|\Pi^{(2)}(1, 2)\rangle = \sum_{ij} a_{ij}\, |\varphi_i(1)\rangle\, |\varphi_j(2)\rangle$, $a_{ij} = -a_{ji}$. A vector $|\phi\rangle = \sum_t b_t|\varphi_t\rangle$ belongs to $V_\perp^{\Pi 1}$ iff it satisfies $\sum_j a_{ij}b_j^* = 0$. If one considers a linear operator A whose representation is given, in the considered basis, by the matrix a_{ij}, then $|\phi\rangle \in V_\perp^{\Pi 1}$ iff A admits the zero eigenvalue, the vector $(b_1^*,\ldots, b_k^*,\ldots)$ being the associated eigenvector. The reader will have no difficulty in realizing that an operator A whose matrix elements satisfy $a_{ij} = -a_{ji}$ and does not admit the zero eigenvalue is easily constructed. Actually the Pauli matrix σ_y is such an operator. The conclusion is that the request—which we will do in what follows—that $V_\perp^{\Pi 1}$ does not reduce to the zero vector, implies by itself some constraints for the state $|\Pi^{(M)}\rangle$. If such constraints are not satisfied, then the procedure we are going to present cannot be developed and the state $|\Pi^{(M)}\rangle$ cannot be combined with another state $|\Phi^{(K)}\rangle$ $(K = N - M)$ to generate a state of $\mathscr{H}_A^{(N)}$ such that there are "subsets" of M and K particles possessing definite properties.

0 being the function of the variables $(2, \ldots, M)$ which vanishes almost everywhere. It goes without saying that, due to the skew-symmetry of $\Pi^{(M)}(1, \ldots, M)$ in its arguments, the same condition (9.8) can be written by saturating an arbitrary variable, i.e., by imposing that $|\Phi^{(1)}\rangle$ satisfies $\int dX\, \Phi^{(1)*}(X)\, \Pi^{(M)}(1, \ldots, X-1, X, X+1, \ldots, M) = 0$.

Since $V_{\perp}^{\Pi 1}$ is a closed linear submanifold of $\mathscr{H}^{(1)}$ we can now consider its orthogonal complement $V^{\Pi 1}$:

$$\mathscr{H}^{(1)} = V^{\Pi 1} \oplus V_{\perp}^{\Pi 1} \tag{9.9}$$

and we can choose a complete orthonormal set $\{|\phi_i\rangle\}$ of single particle states such that, splitting the whole set of nonnegative integers into two disjoint subsets Δ and $\Delta_{\perp}$, one has:

$$|\phi_i\rangle \in V^{\Pi 1} \Leftrightarrow i \in \Delta; \qquad |\phi_i\rangle \in V_{\perp}^{\Pi 1} \Leftrightarrow i \in \Delta_{\perp} \tag{9.10}$$

The following theorem will be useful in what follows:

Theorem 9.1. The vector $|\Pi^{(M)}(1, \ldots, M)\rangle$ can be written as:

$$|\Pi^{(M)}(1, \ldots, M)\rangle = \sum_{i_1 \cdots i_M \in \Delta} c_{i_1 \cdots i_M} |\phi_{i_1}(1)\rangle \cdots |\phi_{i_M}(M)\rangle \tag{9.11}$$

where all the indices $i_1, \ldots, i_M$ belong to Δ and moreover all single particle states whose indices belong to Δ actually appear (in the sense that some nonvanishing coefficients characterized by them occur) at the r.h.s. of the above equation.

In other words, the Fourier expansion of $|\Pi^{(M)}\rangle$ in terms of the states of the basis $\{|\phi_i\rangle\}$ involves all single particle states spanning $V^{\Pi 1}$ and no single particle state spanning $V_{\perp}^{\Pi 1}$.

Proof. Suppose there exists an index k belonging to $\Delta_{\perp}$ such that $c_{ki_2 \cdots i_M} \neq 0$ for at least one choice of the indices $i_2 \cdots i_M$. On the other hand, since $k \in \Delta_{\perp}$, the single particle basis vector $|\phi_k\rangle$ belongs to $V_{\perp}^{\Pi 1}$ and, as such, it satisfies

$$\langle \phi_k \mid \Pi^{(M)} \rangle = \sum_{i_2 \cdots i_M} c_{ki_2 \cdots i_M} |\phi_{i_2}(2)\rangle \cdots |\phi_{i_M}(M)\rangle = 0 \tag{9.12}$$

Since the vectors $|\phi_{i_2}(2)\rangle \cdots |\phi_{i_M}(M)\rangle$ are linearly independent for any given choice of the indices $i_2 \cdots i_M$, Eq. (9.12) implies $c_{ki_2 \cdots i_M} = 0$, $\forall i_2 \cdots i_M$, which is contrary to the hypothesis.

On the other hand, let us suppose that there exists an index j belonging to $\varDelta$ such that $c_{ji_2\cdots i_M} = 0\ \forall i_2 \cdots i_M$. This means that

$$\langle \phi_j \mid \varPi^{(M)} \rangle = \sum_{i_2 \cdots i_M} c_{ji_2\cdots i_M} \,|\phi_{i_2}(2)\rangle \cdots |\phi_{i_M}(M)\rangle = 0 \tag{9.13}$$

implying that the vector $|\phi_j\rangle$ belongs to $V_{\perp}^{\varPi 1}$, which is absurd. ∎

Summarizing, choosing any vector $|\varPi^{(M)}(1,..., M)\rangle$ of $\mathscr{H}_A^{(M)}$ such that $V_{\perp}^{\varPi 1}$ differs from the zero vector of $\mathscr{H}^{(1)}$, uniquely identifies two closed linear submanifolds of $\mathscr{H}^{(1)}$ whose direct sum coincides with $\mathscr{H}^{(1)}$ itself, and, correspondingly, a complete orthonormal set of single particle states which is the union of two subsets $\{|\phi_i\rangle\}$, $i \in \varDelta$ and $\{|\phi_j\rangle\}$, $j \in \varDelta_{\perp}$ such that all and only the states $\{|\phi_i\rangle\}$ for which $i \in \varDelta$ enter into the Fourier expansion of $|\varPi^{(M)}(1,..., M)\rangle$ in terms of the basis generated by the antisymmetrized and normalized products of the set $\{|\phi_i\rangle\}$.

We pass now from the Hilbert space $\mathscr{H}^{(1)}$ to the spaces $\mathscr{H}_A^{(M)}$ and $\mathscr{H}_A^{(K)}$, which, as already stated, are those we are interested in. Having partitioned the complete set of single particle states $\{|\phi_j\rangle\}$ into two subsets according to their indices belonging to $\varDelta$ or $\varDelta_{\perp}$, we consider now two important (for our purposes) proper submanifolds $V^{\varPi M}$ of $\mathscr{H}_A^{(M)}$ and $V_{\perp}^{\varPi K}$ of $\mathscr{H}_A^{(K)}$, respectively. They are simply the manifolds spanned by the states:

$$\begin{aligned} V^{\varPi M}&: \quad \frac{1}{\sqrt{M!}} A\{|\phi_{i_1}\rangle,..., |\phi_{i_M}\rangle\}, \qquad i_1,..., i_M \in \varDelta \\ V_{\perp}^{\varPi K}&: \quad \frac{1}{\sqrt{K!}} A\{|\phi_{j_1}\rangle,..., |\phi_{j_K}\rangle\}, \qquad j_1,..., j_K \in \varDelta_{\perp} \end{aligned} \tag{9.14}$$

In brief, $V^{\varPi M}$ is the set of all the states of $\mathscr{H}_A^{(M)}$ such that their Fourier expansion in terms of the single particle states $\{|\phi_i\rangle\}$ contains only states whose indices belong to $\varDelta$, and $V_{\perp}^{\varPi K}$ is the set of all states of $\mathscr{H}_A^{(K)}$ such that their Fourier expansions contains only states whose indices belong to $\varDelta_{\perp}$.

In virtue of our definition of the two manifolds $V^{\varPi M}$ and $V_{\perp}^{\varPi K}$, there follows trivially that the saturation of any variable of a state $|\varPhi^{(K)}\rangle \in V_{\perp}^{\varPi K}$ with any variable of a state $|\varSigma^{(M)}\rangle \in V^{\varPi M}$ gives the null function of the unsaturated variables:

$$\int dX\, \varSigma^{(M)}(1,..., X,...M)\, \varPhi^{(K)*}(M+1,..., X,...N) = 0 \tag{9.15}$$

We will call a pair of states for which (9.15) holds “one-particle orthogonal.” Analogously, when we have two closed linear manifolds such that condition (9.15) is satisfied for any pair of vectors taken from one and the other of them, we will say that the manifolds themselves are "one-particle orthogonal."

With reference to Eq. (9.15) we would like to call attention to the fact that, taking into account the preceding arguments, one can easily prove the following theorem:

Theorem 9.2. Given any pair of states which are "one-particle orthogonal," one can find an appropriate complete orthonormal single particle basis such that the Fourier expansions of the two states involve disjoint subsets of the states of this single particle basis.

The proof is easily obtained by noticing, first of all, that Eq. (9.15), if one fixes the value of all the variables appearing in $\Phi^{(K)}$ different from X, shows that the manifold $V_{\perp}^{\Sigma 1}$ (with obvious meaning of the symbol) does not reduce to the zero vector. One then can follow the previous procedure to build the appropriate single particle basis satisfying the above theorem.

9.1.2. Antisymmetrized Products of Appropriate States of $\mathscr{H}_A^{(K)}$ and a Given State of $\mathscr{H}_A^{(M)}$

In this subsection we consider a fixed state $|\Pi^{(M)}(1,..., M)\rangle$ of $\mathscr{H}_A^{(M)}$ and an arbitrary state $|\Phi^{(K)}(M+1,..., N)\rangle$ of $V_{\perp}^{\Pi K}$. We take the direct product of the two and we totally antisymmetrize it, i.e., we consider the non-normalized state

$$
\begin{aligned}
|\tilde{\psi}^{(N)}(1,..., N)\rangle &= P_A[|\Pi^{(M)}(1,..., M)\rangle \otimes |\Phi^{(K)}(M+1,..., N)\rangle] \\
&= \frac{1}{N!} A[|\Pi^{(M)}(1,..., M)\rangle \otimes |\Phi^{(K)}(M+1,..., N)\rangle] \qquad (9.16)
\end{aligned}
$$

where the linear operator $P_A = \frac{1}{N!} A$ is the projection operator on the submanifold $\mathscr{H}_A^{(N)}$ of $\mathscr{H}^{(1)} \otimes \cdots \otimes \mathscr{H}^{(1)}$.

To evaluate its norm as well as to prove a theorem which will be useful in what follows, it is convenient to resort to a simple trick by dividing the permutations of the N particles implied by the symbol A in the above equation, into two families $\mathscr{F}$ and $\mathscr{G}$, where $\mathscr{F}$ contains all the permutations which exchange the first M and/or the second K variables among themselves, while $\mathscr{G}$ contains only permutations which exchange *at least* one variable $(1,..., M)$ with the remaining ones. It holds :

$$
A \equiv \left[\sum_{\mathscr{F}} (-1)^f F + \sum_{\mathscr{G}} (-1)^g G \right] \qquad (9.17)
$$

f and g being the parity of the corresponding permutations.

Note that, since

$$
\begin{aligned}
&F[|\Pi^{(M)}(1,\ldots, M)\rangle \otimes |\Phi^{(K)}(M+1,\ldots, N)\rangle] \\
&\qquad = (-1)^f\, [|\Pi^{(M)}(1,\ldots, M)\rangle \otimes |\Phi^{(K)}(M+1,\ldots, N)\rangle]
\end{aligned} \tag{9.18}
$$

we have

$$
\begin{aligned}
&\sum_{\mathscr{F}} (-1)^f\, F[|\Pi^{(M)}(1,\ldots, M)\rangle \otimes |\Phi^{(K)}(M+1,\ldots, N)\rangle] \\
&\qquad = \sum_{\mathscr{F}}\, [|\Pi^{(M)}(1,\ldots, M)\rangle \otimes |\Phi^{(K)}(M+1,\ldots, N)\rangle] \\
&\qquad = M!\, K!\, [|\Pi^{(M)}(1,\ldots, M)\rangle \otimes |\Phi^{(K)}(M+1,\ldots, N)\rangle]
\end{aligned} \tag{9.19}
$$

Before going on we remark that if $|\chi^{(K)}\rangle$ belongs to $V_{\perp}^{\Pi K}$ one has

$$
\langle \chi^{(K)}(M+1,\ldots, N)|\sum_{\mathscr{G}} (-1)^g\, G\, |\Pi^{(M)}(1,\ldots, M)\rangle\, |\Phi^{(K)}(M+1,\ldots, N)\rangle = 0 \tag{9.20}
$$

where zero denotes the function of the variables $(1,\ldots, M)$ which vanishes almost everywhere. In fact any individual term of the sum over $\mathscr{G}$ has at least one of the variables from $M+1$ to N which belongs to the state $|\Pi^{(M)}\rangle$ and, as such, it involves single particle state indices confined to the set Δ. Since the same variable belongs to the state $|\chi^{(K)}\rangle$, and therefore it is associated to single particle states whose indices belong to $\Delta_{\perp}$, the integration over such a variable gives the result zero.

Coming back to our unnormalized state (9.16), taking into account that $P_A^2 = P_A$, we have

$$
\begin{aligned}
&\langle \tilde{\psi}^{(N)} \,|\, \tilde{\psi}^{(N)}\rangle \\
&= \frac{1}{N!} \langle \Pi^{(M)}(1,\ldots, M)\, \Phi^{(K)}(M+1,\ldots, N)| \\
&\qquad \times \left[\sum_{\mathscr{F}} (-1)^f\, F + \sum_{\mathscr{G}} (-1)^g\, G \right] |\Pi^{(M)}(1,\ldots, M)\, \Phi^{(K)}(M+1,\ldots, N)\rangle \\
&= \frac{1}{N!} \sum_{\mathscr{F}} \langle \Pi^{(M)}(1,\ldots, M)\, \Phi^{(K)}(M+1,\ldots, N) \,|\, \Pi^{(M)}(1,\ldots, M)\, \Phi^{(K)}(M+1,\ldots, N)\rangle \\
&= \frac{K!\, M!}{N!}
\end{aligned} \tag{9.21}
$$

In deriving the above equation we have taken into account the fact that, since $|\Phi^{(K)}(M+1,\ldots, N)\rangle \in V_{\perp}^{\Pi K}$, the sum over $\mathscr{G}$ does not contribute, in accordance with (9.20). The correctly normalized state we are interested in is then

$$|\psi^{(N)}(1,\ldots, N)\rangle = \sqrt{\binom{N}{K}}\, P_A[|\Pi^{(M)}(1,\ldots, M)\rangle \otimes |\Phi^{(K)}(M+1,\ldots, N)\rangle] \tag{9.22}$$

Having identified the closed linear manifold $V_{\perp}^{\Pi K}$ of $\mathscr{H}_A^{(K)}$ we consider now a complete orthonormal set $\{|\Theta_{\perp i}^{(K)}(M+1,\ldots, N)\rangle\}$ which spans such a manifold. In terms of these states and of the state $|\Pi^{(M)}(1,\ldots, M)\rangle$ we build up the orthonormal set $\{|\omega_{\perp i}^{(N)}(1,\ldots, N)\rangle\}$ of states of $\mathscr{H}_A^{(N)}$ according to:

$$|\omega_{\perp i}^{(N)}(1,\ldots, N)\rangle \equiv \sqrt{\binom{N}{K}}\, P_A[|\Pi^{(M)}(1,\ldots, M)\rangle \otimes |\Theta_{\perp i}^{(K)}(M+1,\ldots, N)\rangle] \tag{9.23}$$

We already know that such states are normalized, while their orthogonality is easily proved by taking into account that $P_A^2 = P_A$, and Eqs. (9.17) and (9.20).

There follows that the operators $|\omega_{\perp i}^{(N)}\rangle\langle\omega_{\perp i}^{(N)}|$ are a set of orthogonal projection operators and, consequently, the operator

$$E_{A\perp}^{\Pi(N)} = \sum_i |\omega_{\perp i}^{(N)}\rangle\langle\omega_{\perp i}^{(N)}| \tag{9.24}$$

is also a projection operator of $\mathscr{H}_A^{(N)}$.

9.1.3. Some Useful Technical Details about the Formal Procedure of the Previous Subsections

The identification of the "one-particle orthogonal" linear manifolds $V^{\Pi M}$ and $V_{\perp}^{\Pi K}$ has been made starting from the consideration of a precise state $|\Pi^{(M)}\rangle$ of $\mathscr{H}_A^{(M)}$. However, since we will be interested in states like (9.22) which are obtained by antisymmetrizing a product of a state $|\Pi^{(M)}\rangle$ and a state $|\Phi^{(K)}\rangle$ which are "one-particle orthogonal," we could have followed the opposite line of approach, by assigning the prominent role to the state $|\Phi^{(K)}\rangle$. In doing this we would have been led to identify two "one-particle orthogonal" linear manifolds $V^{\Phi K}$ and $V_{\perp}^{\Phi M}$, which differ, in general, from those mentioned above.

Note, however, that just as $|\Pi^{(M)}\rangle \in V^{\Pi M}$ and $|\Phi^{(K)}\rangle \in V_{\perp}^{\Pi K}$, it also happens that $|\Pi^{(M)}\rangle \in V_{\perp}^{\Phi M}$ and $|\Phi^{(K)}\rangle \in V^{\Phi K}$.

The simplest example of the above situation is represented, in the case of two identical fermions, by the state:

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2}}\left[|\Lambda(1)\rangle\,|\Gamma(2)\rangle - |\Gamma(1)\rangle\,|\Lambda(2)\rangle\right] \tag{9.25}$$

with $\langle\Lambda(i)\,|\,\Gamma(i)\rangle = 0$. In such a case, if we start with the state $|\Lambda\rangle$, the manifold $V_{\perp}^{\Lambda 1}$ is the one spanned by $|\Gamma\rangle$ and by an orthonormal set of states $|\Theta_{\perp i}\rangle$ which spans the manifold orthogonal to both $|\Gamma\rangle$ and $|\Lambda\rangle$. If we identify the manifolds by the corresponding projection operators, we have:

$$\begin{aligned} V^{\Lambda 1} &\Leftrightarrow P = |\Lambda\rangle\langle\Lambda| \\ V_{\perp}^{\Lambda 1} &\Leftrightarrow P = |\Gamma\rangle\langle\Gamma| + \sum_i |\Theta_{\perp i}\rangle\langle\Theta_{\perp i}| \end{aligned} \tag{9.26}$$

On the contrary, if we choose the state $|\Gamma\rangle$ to set up our procedure, we would end up with the two manifolds:

$$\begin{aligned} V^{\Gamma 1} &\Leftrightarrow P = |\Gamma\rangle\langle\Gamma| \\ V_{\perp}^{\Gamma 1} &\Leftrightarrow P = |\Lambda\rangle\langle\Lambda| + \sum_i |\Theta_{\perp i}\rangle\langle\Theta_{\perp i}| \end{aligned} \tag{9.27}$$

It goes without saying that in such a case one could consider two other one particle orthogonal manifolds W and $W_{\perp}$ as follows:

$$\begin{aligned} W &\Leftrightarrow P = |\Lambda\rangle\langle\Lambda| + \sum_{i\in\delta} |\Theta_{\perp i}\rangle\langle\Theta_{\perp i}| \\ W_{\perp} &\Leftrightarrow P = |\Gamma\rangle\langle\Gamma| + \sum_{i\in\delta'} |\Theta_{\perp i}\rangle\langle\Theta_{\perp i}| \end{aligned} \tag{9.28}$$

δ and δ' representing a partition of the positive integers.

It is very easy to understand the formal reasons of the just considered situation. When dealing with a system of N fermions, we started with the state $|\Pi^{(M)}\rangle$, and we have identified the single particle submanifold $V_{\perp}^{\Pi 1}$ characterized by the orthonormal set $|\phi_j\rangle$, $j \in \Delta_{\perp}$. We have then defined Δ as the complement of $\Delta_{\perp}$ and we have shown that the Fourier expansion of $|\Pi^{(M)}\rangle$ involves all single particle states $|\phi_i\rangle$ for which $i \in \Delta$. We have also taken into account a state $|\Phi^{(K)}\rangle \in V_{\perp}^{\Pi K}$. We remark now that there is no reason why the Fourier expansion of $|\Phi^{(K)}\rangle$ should involve all states for which $j \in \Delta_{\perp}$. Suppose it actually involves only a subset Δ_{Φ} of $\Delta_{\perp}$. If this is the case, we are naturally led to consider the following eigenmanifolds:

- the manifold $V_{\Delta}^{(M)}$ which is spanned by the basis vectors

$$\frac{1}{\sqrt{M!}} A\{|\phi_{i_1}\rangle, \ldots, |\phi_{i_M}\rangle\}, \qquad i_1, \ldots, i_M \in \Delta \tag{9.29}$$

- the manifold $V_{\Delta_\Phi}^{(K)}$ which is spanned by the basis vectors

$$\frac{1}{\sqrt{K!}} A\{|\phi_{j_1}\rangle, \ldots, |\phi_{j_K}\rangle\}, \qquad j_1, \ldots, j_K \in \Delta_\Phi \subset \Delta_\perp \tag{9.30}$$

- the manifolds $V_{\Delta'}^{(M)}$ and $V_{\Delta'}^{(K)}$ which are spanned by the basis vectors

$$\frac{1}{\sqrt{M!}} A\{|\phi_{r_1}\rangle, \ldots, |\phi_{r_M}\rangle\}, \qquad r_1, \ldots, r_M \in \Delta' \tag{9.31}$$

and

$$\frac{1}{\sqrt{K!}} A\{|\phi_{s_1}\rangle, \ldots, |\phi_{s_K}\rangle\}, \qquad s_1, \ldots, s_K \in \Delta' \tag{9.32}$$

respectively, where Δ' contains all single particle indices which do not belong to Δ or Δ_Φ.

Despite the fact that there seems to be a certain degree of freedom in choosing a couple of "one-particle orthogonal" manifolds, the appropriateness of the above remarks will appear clearly when, in Section 9.2.4, we will discuss the physical meaning of our requirements concerning complete sets of properties and/or the non-entangled character of appropriate subsets of a system of identical constituents. We will in fact show that the one particle orthogonality is a necessary condition in order that one can do the physics within each such manifold by disregarding the other.

9.2. Entanglement and Properties for Systems of *N* Identical Fermions

Bearing in mind the mathematical formalism we have introduced in the previous sections, we can now formalize the idea of non-entangled states of a system of identical fermions.

9.2.1. States of Many Identical Fermions and Their Properties

We begin by characterizing in a mathematically precise way the fact that a subgroup, or, better a "subquaset" of M of the N identical fermions possesses a complete set of properties:

Definition 9.1. Given a system $S^{(N)}$ of N identical fermions in a pure state $|\psi^{(N)}\rangle$ of $\mathscr{H}_A^{(N)}$ we will claim that two subsets of cardinality M and K ($N = M + K$), respectively, both possess a complete set of properties iff there exists a state $|\Pi^{(M)}\rangle$ of $\mathscr{H}_A^{(M)}$ such that

$$\langle\psi^{(N)}|\, E_{A\perp}^{\Pi(N)}(1,..., N)\, |\psi^{(N)}\rangle = 1 \tag{9.33}$$

where $E_{A\perp}^{\Pi(N)}(1,..., N) = \sum_s |\omega_{\perp s}^{(N)}(N)\rangle\langle\omega_{\perp s}^{(N)}|$ is the projection operator of $\mathscr{H}_A^{(N)}$ given by Eq. (9.24), which is uniquely identified, according to the previous procedure, by the assignment of the state $|\Pi^{(M)}\rangle$.

Condition (9.33) assures that, given the state $|\psi^{(N)}\rangle$ of N identical fermions and the specific single particle basis $\{|\phi_k\rangle\}$ of Eq. (9.10), the probability of finding "a group" of M particles described by the state $|\Pi^{(M)}\rangle$ *and* the remaining ones in single particle states which do not appear in the Fourier decomposition of $|\Pi^{(M)}\rangle$, equals one. In the next subsections we will discuss the precise physical reasons for which, when the above conditions are satisfied, it is correct to claim that "there is a set of M fermions which is non entangled with the remaining set of K particles."

Here we note the significant fact that $E_{A\perp}^{\Pi(N)}(1,..., N)$ turns out to be the restriction to the totally antisymmetric manifold of N indistinguishable fermions of the projection operator $E(1,..., N) = \tilde{S}[|\Pi^{(M)}\rangle\langle\Pi^{(M)}| \otimes \sum_i |\Theta_{\perp i}^{(K)}\rangle\langle\Theta_{\perp i}^{(K)}|]$, where we have indicated with $\tilde{S}$ the sum of the $\binom{N}{K} = \binom{N}{M}$ terms in which one or more of the first M indices are exchanged with the remaining ones.

In fact, by resorting to the projection operator $P_A = \frac{1}{N!}A$ onto the manifold $\mathscr{H}_A^{(N)}$, we have:

$$\begin{aligned}
P_A E(1,..., N)\, P_A &= \frac{1}{(N!)^2} A \Bigg[\tilde{S} \Bigg[|\Pi(1 \cdots M)\rangle\langle\Pi(1 \cdots M)| \\
&\quad \otimes \sum_i |\Theta_{\perp i}(M+1 \cdots N)\rangle\langle\Theta_{\perp i}(M+1 \cdots N)| \Bigg]\Bigg] A \\
&= \frac{1}{(N!)^2}\binom{N}{K} \sum_i A[|\Pi(1 \cdots M)\rangle\, |\Theta_{\perp i}(M+1 \cdots N)\rangle] \\
&\quad \times [\langle\Pi(1 \cdots M)|\, \langle\Theta_{\perp i}(M+1 \cdots N)|]\, A \\
&= \sum_i |\omega_{\perp i}^{(N)}\rangle\langle\omega_{\perp i}^{(N)}| \equiv E_{A\perp}^{\Pi(N)}
\end{aligned} \tag{9.34}$$

where the coefficient $\binom{N}{K}$ corresponds to the number of permutations produced by the operator $\tilde{S}$, while the last line follows directly from the definition of the states (9.23).

In the special case of two identical fermions, we notice that the operator $E_{A\perp}^{\Pi(2)} = \sum_i |\omega_{\perp i}^{(2)}\rangle\langle\omega_{\perp i}^{(2)}|$ reduces to the one of Eq. (7.3) with $P^{(1)} \otimes P^{(2)} = 0$, which we have used to define the properties of two fermion states.

In fact, if we denote with P the projection operator of $\mathscr{H}^{(1)}$ which projects onto the one-dimensional manifold spanned by the state $|\Pi^{(1)}\rangle$, it is easy to see that $E_{A\perp}^{\Pi(2)}$ turns out to be:

$$\begin{aligned} E_{A\perp}^{\Pi(2)} &= \sum_i |\omega_{\perp i}^{(2)}\rangle\langle\omega_{\perp i}^{(2)}| \\ &= \tfrac{1}{2} P^{(1)} \otimes (I^{(2)} - P^{(2)}) + \tfrac{1}{2}(I^{(1)} - P^{(1)}) \otimes P^{(2)} \\ &\quad - \tfrac{1}{2} \sum_i [|\Theta_{\perp i}(1)\rangle\, |\Pi(2)\rangle - |\Pi(1)\rangle\, |\Theta_{\perp i}(2)\rangle] \end{aligned} \tag{9.35}$$

Since it is easy to prove that the last term, when acting on $\mathscr{H}_A^{(2)}$, coincides with the sum of the first two terms, we have proved that:

$$E_{A\perp}^{\Pi(2)} = (I^{(1)} - P^{(1)}) \otimes P^{(2)} + P^{(1)} \otimes (I^{(2)} - P^{(2)}) \tag{9.36}$$

Equation (9.36), shows that the general formalism we have developed in order to deal with the properties of N identical fermions reduces to the one we have already used in Section 7 when dealing with two identical quantum constituents of a composite system.

9.2.2. A Relevant Theorem

It is now extremely easy to prove the following theorem which identifies the mathematical properties to be satisfied by a state vector in order that it describes subsets of identical constituents possessing a complete set of properties.

Theorem 9.3. A necessary and sufficient condition in order that a state $|\psi^{(N)}\rangle$ of the Hilbert space of N identical fermions allows the identification of two subsets of cardinality M and K ($N = M + K$) of particles which possess a complete set of properties is that $|\psi^{(N)}\rangle$ be obtained by antisymmetrizing and normalizing the direct product of two states $|\Pi^{(M)}(1,\dots, M)\rangle$ and $|\Phi^{(K)}(M+1,\dots, N)\rangle$ of $\mathscr{H}_A^{(M)}$ and $\mathscr{H}_A^{(K)}$ respectively, where $|\Phi^{(K)}(M+1,\dots, N)\rangle$ belongs to $V_\perp^{\Pi K}$.

Proof. Suppose that:

$$|\psi^{(N)}\rangle = \sqrt{\binom{N}{K}}\, P_A[|\Pi^{(M)}(1,\dots, M)\rangle \otimes |\Phi^{(K)}(M+1,\dots, N)\rangle] \tag{9.37}$$

with $|\Phi^{(K)}(M+1,\ldots,N)\rangle \in V_{\perp}^{\Pi K}$. Then one can write:

$$|\Phi^{(K)}(M+1,\ldots,N)\rangle = \sum_s a_s \, |\Theta_{\perp s}^{(K)}(M+1,\ldots,N)\rangle \tag{9.38}$$

Replacing (9.38) in (9.37) one has:

$$\begin{aligned} |\psi^{(N)}\rangle &= \sqrt{\binom{N}{K}} \sum_s a_s P_A[|\Pi^{(M)}(1,\ldots,M)\rangle \otimes |\Theta_{\perp s}^{(K)}(M+1,\ldots,N)\rangle] \\ &= \sum_s a_s \, |\omega_{\perp s}^{(N)}\rangle \end{aligned} \tag{9.39}$$

There follows:

$$E_{A\perp}^{\Pi(N)} \, |\psi^{(N)}\rangle = \sum_{rs} a_s \, |\omega_{\perp r}^{(N)}\rangle\langle\omega_{\perp r}^{(N)} \mid \omega_{\perp s}^{(N)}\rangle = \sum_s a_s \, |\omega_{\perp s}^{(N)}\rangle = |\psi^{(N)}\rangle \tag{9.40}$$

which implies

$$\langle\psi^{(N)}| \, E_{A\perp}^{\Pi(N)} \, |\psi^{(N)}\rangle = 1 \tag{9.41}$$

Conversely, since $E_{A\perp}^{\Pi(N)}$ is a projection operator, the condition $\langle\psi^{(N)}| \, E_{A\perp}^{\Pi(N)} \, |\psi^{(N)}\rangle = 1$ implies:

$$E_{A\perp}^{\Pi(N)} \, |\psi^{(N)}\rangle = |\psi^{(N)}\rangle \tag{9.42}$$

i.e., putting $b_s = \langle\omega_{\perp s}^{(N)} \mid \psi^{(N)}\rangle$:

$$\begin{aligned} |\psi^{(N)}\rangle &= \sum_s |\omega_{\perp s}^{(N)}\rangle\langle\omega_{\perp s}^{(N)} \mid \psi^{(N)}\rangle \\ &= \sqrt{\binom{N}{K}} P_A \sum_s b_s[|\Pi^{(M)}\rangle \otimes |\Theta_{\perp s}^{(K)}\rangle] \\ &= \sqrt{\binom{N}{K}} P_A \left[|\Pi^{(M)}\rangle \otimes \sum_s b_s \, |\Theta_{\perp s}^{(K)}\rangle \right] \\ &= \sqrt{\binom{N}{K}} P_A[|\Pi^{(M)}\rangle \otimes |\Phi^{(K)}\rangle] \end{aligned} \tag{9.43}$$

where $|\Phi^{(K)}\rangle = \sum_s b_s \, |\Theta_{\perp s}^{(K)}\rangle \in V_{\perp}^{\Pi K}$. ∎

9.2.3. Non-Entangled Subsets of N Identical Fermions

As we have just shown, the requirement that a "subset" of M fermions of a system of N identical fermions in the pure state $|\psi^{(N)}\rangle$ possesses a complete set of properties requires that the state vector of the system has the form (9.37) and this happens iff Eq. (9.33) is satisfied. We now note that, in the case under consideration, the two "factors" $|\Pi^{(M)}\rangle$ and $|\Phi^{(K)}\rangle$, play a perfectly symmetrical role. It follows that it is possible to define a projection operator $E_{A\perp}^{\Phi N}$, which plays precisely the same role as $E_{A\perp}^{\Pi N}$ and, correspondingly it allows us to claim that "a subset of K fermions has the complete set of properties associated to the pure state $|\Phi^{(K)}(M+1,..., N)\rangle$."

Since, according to Definition 8.1, we have related the possibility of claiming that the system of N fermions contains two non-entangled "subsets" of cardinality M and K, respectively, to the possibility of attributing a complete set of properties to the two subsets under examination, the following theorem has been proved:

Theorem 9.4. Given a system $S^{(N)}$ of N identical fermions in a pure state $|\psi^{(N)}\rangle$ of $\mathscr{H}_A^{(N)}$ it contains two non-entangled subsets of cardinality M and K iff $|\psi^{(N)}\rangle$ can be written as:

$$|\psi^{(N)}\rangle = \sqrt{\binom{N}{K}}\, P_A[|\Pi^{(M)}(1,..., M)\rangle \otimes |\Phi^{(K)}(M+1,..., N)\rangle] \tag{9.44}$$

where the states $|\Pi^{(M)}(1,..., M)\rangle$ and $|\Phi^{(K)}(M+1,..., N)\rangle$ are "one-particle orthogonal" among themselves.

9.2.4. The Physical Motivations for the Conditions that a Subset Be Non-Entangled

In this section we present precise physical arguments which should clarify the formal constraints we have imposed to claim that a subset of a set of identical particles is non-entangled with the remaining ones, in spite of the fact that the whole state satisfies the requirement of being totally antisymmetric. To this purpose, let us take into account a complete orthonormal set of single particle states and let us consider, in strict analogy with what we have done in Section 9.1.3, two disjoint subsets[20] Δ and Δ^*. We

[20] Note that we do not assume that the union of the states with indices belonging to such subsets coincides with the complete orthonormal single particle basis. As a clarifying example one can consider two denumerable set of states spanning the manifold of the square integrable functions with disjoint supports A and B, respectively, such that $A \cup B$ is strictly contained in $\mathbf{R}^3$.

consider, as usual, the two closed submanifolds $V_{\Delta}^{(M)}$ and $V_{\Delta^*}^{(K)}$ of $\mathscr{H}_{A}^{(M)}$ and $\mathscr{H}_{A}^{(K)}$, respectively, with obvious meaning of the symbols: they are the manifolds such that the Fourier development of their states involve only single particle states whose indices belong to Δ, or to Δ^*, respectively. In place of a precise state $|\Pi^{(M)}\rangle$ (as we did up to now), here we consider arbitrary states of $V_{\Delta}^{(M)}$ and $V_{\Delta^*}^{(K)}$, and two orthonormal bases $\{|\Upsilon_j^{(M)}\rangle\}$ and $\{|\Xi_l^{(K)}\rangle\}$ spanning such manifolds.

By repeating the calculations of the previous sections, one immediately proves that, if consideration is given to an arbitrary pair of states $|\chi^{(M)}\rangle$ and $|\tau^{(M)}\rangle$ of $V_{\Delta}^{(M)}$ or to another pair of states $|\mu^{(K)}\rangle$ and $|\nu^{(K)}\rangle$ of $V_{\Delta^*}^{(K)}$, the following relations hold:

$$\sum_l \left\| \left[\sqrt{\binom{N}{K}} \langle\chi^{(M)}| \langle\Xi_l^{(K)}| P_A \right] \cdot \left[\sqrt{\binom{N}{K}} P_A |\tau^{(M)}\rangle |\nu^{(K)}\rangle \right] \right\|^2 = |\langle\chi^{(M)} | \tau^{(M)}\rangle|^2 \tag{9.45}$$

and

$$\sum_j \left\| \left[\sqrt{\binom{N}{K}} \langle\Upsilon_j^{(M)}| \langle\mu^{(K)}| P_A \right] \cdot \left[\sqrt{\binom{N}{K}} P_A |\tau^{(M)}\rangle |\nu^{(K)}\rangle \right] \right\|^2 = |\langle\mu^{(K)} | \nu^{(K)}\rangle|^2 \tag{9.46}$$

These equations show that, provided two one-particle orthogonal manifolds $V_{\Delta}^{(M)}$ and $V_{\Delta^*}^{(K)}$ can be identified, and provided the interactions between the particles determining the subsequent evolution do not alter the specific features of the state vector, then one can *do the physics within each manifold by disregarding the other one*, even though the appropriate antisymmetrization requests for the whole set of fermions are respected.[21] These considerations should have made clear the appropriateness of adopting our criteria (as given in Theorem 9.4) for the attribution of complete sets of properties associated to $|\Pi^{(M)}\rangle$ and $|\Phi^{(K)}\rangle$ and for the identification of non-entangled subsets of a system of N identical fermions.

A concluding remark. Obviously, (see also the discussion by A. Messiah in his book[(21)]), the most significant instance of the above situation is the one in which the set of states whose indices belong to Δ are single particle states whose wave functions have compact support within a region A, while those whose indices belong to Δ^* have support confined to a region B disjoint from A. In such a case we can claim that "there are M

[21] With reference to Eq. (9.45), we stress that the one-particle orthogonality of the states $|\tau^{(M)}\rangle$ and $|\nu^{(K)}\rangle$, and $|\chi^{(M)}\rangle$ and $|\Xi^{(K)}\rangle$ as well as the corresponding ones for the states appearing in Eq. (9.46), is absolutely fundamental—as the reader can check—in order that the (physically important) equality sign between the expressions at the left and right hand sides of the equations hold.

fermions in region A with precise properties," in spite of the fact that the presence of the remaining identical fermions (confined within a different region) has been rigorously taken into account.

The particular relevance of the case in which the "subsets" of our analysis are related to single particle states with disjoint supports, emerges clearly from the previous considerations. The special situation according to which we can "look at a part of the universe" disregarding the rest of it (which however has precise implications for the total state vector) occurs easily (at least to an extremely high degree of accuracy—see the next section) when the manifolds $V_{\Delta}^{(M)}$ and $V_{\Delta^*}^{(K)}$ correspond precisely to single particle states with compact disjoint supports.

With reference to this fact, we point out that in this paper we have dealt with the general case and we have avoided to put, from the very beginning, limitations to the specific structure of our one-particle orthogonal manifolds, to stick as far as possible to a rigorous and general mathematical treatment. However, we can now call attention to the fact that if the identification of the two considered manifolds would be related, e.g., to the indices Δ and Δ^* corresponding to single particle states having different parity under space reflections, then, even though for a N-particle state like $\sqrt{\binom{N}{K}}\, P_A\, |\Pi^{(M)}\rangle\, |\Phi^{(K)}\rangle$ (which is built in terms of a pair of one-particle orthogonal states $|\Pi^{(M)}\rangle$ and $|\Phi^{(K)}\rangle$) we can attribute complete sets of properties to the appropriate subsets of cardinality M and K, almost every interaction between the particles will destroy the "factorizability" as well as the one-particle orthogonality of the factors of the complete state.

Once more, as we have repeatedly outlined in this paper, particle positions play an absolutely prominent role in making physically interesting and meaningful our analysis.

The above considerations should have clarified our line of thought in approaching the problem of identifying non-entangled states in the case of systems with identical constituents.

9.2.5. An Important Specification about Non-Entangled States

Our definition of non-entangled subsets of systems of identical fermions might be considered not fully appropriate by some readers, due to the fact that it does not imply the *local* factorizability of position probabilities.[22] To discuss this question it seems useful to limit, for the moment, our considerations to the system of two identical fermions and to derive two simple theorems. We consider a single particle complete othonormal set $\{|\varphi_i\rangle\}$ and two disjoint subsets Δ and Δ^* of the ensemble $\mathbf{N}$ of the

[22] We are grateful to the refereee for having called our attention on the fact that this point deserves a detailed analysis.

natural numbers. We do not require that the union of Δ and Δ^* exhausts the whole $\mathbf{N}$. We denote as V_Δ and V_{Δ^*} the two orthogonal submanifolds of the single particle Hilbert space $\mathscr{H}$ spanned by the vectors $\{|\varphi_i\rangle\}$, with $i \in \Delta$, and $\{|\varphi_j\rangle\}$, with $j \in \Delta^*$, respectively. We consider two orthogonal projection operators P_δ and Q_{δ^*} onto two submanifolds $V_\delta \subseteq V_\Delta$ and $V_{\delta^*} \subseteq V_{\Delta^*}$ of $\mathscr{H}$, and, in terms of them we define the following projection operator:

$$E(1, 2) = P_\delta(1) \otimes Q_{\delta^*}(2) + Q_{\delta^*}(1) \otimes P_\delta(2) \tag{9.47}$$

of the Hilbert space of the system of two identical fermions. Finally we take into account a non-entangled (according to our definition) state vector of our two identical fermions:

$$|\psi(1, 2)\rangle = \frac{1}{\sqrt{2}}[|\Pi(1)\rangle\, |\Phi(2)\rangle - |\Phi(1)\rangle\, |\Pi(2)] \tag{9.48}$$

where $|\Pi\rangle$ belongs to V_Δ and $|\Phi\rangle$ belongs to V_{Δ^*}. With these premises we can now formulate the following theorem:

Theorem 9.5. Given the state (9.48), the joint probability distribution of finding one particle in a state belonging to V_δ and one particle in a state belonging to V_{δ^*} factorizes into the product of the probabilities of the single events.

Proof. The proof is straightforward. According to our analysis of Section 7.1 it amounts simply to verify that:

$$\langle\psi(1, 2)|\, E(1, 2)\, |\psi(1, 2)\rangle = \langle\Pi|\, P_\delta\, |\Pi\rangle \cdot \langle\Phi|\, Q_{\delta^*}\, |\Phi\rangle \tag{9.49}$$

where

$$\begin{aligned}\langle\Pi|\, P_\delta\, |\Pi\rangle = \langle\psi(1, 2)|\, & P_\delta(1) \otimes (I(2) - P_\delta(2)) \\ & + (I(1) - P_\delta(1)) \otimes P_\delta(2)\, |\psi(1, 2)\rangle \end{aligned} \tag{9.50}$$

$$\begin{aligned}\langle\Phi|\, Q_{\delta^*}\, |\Phi\rangle = \langle\psi(1, 2)|\, & Q_{\delta^*}(1) \otimes (I(2) - Q_{\delta^*}(2)) \\ & + (I(1) - Q_{\delta^*}(1)) \otimes Q_{\delta^*}(2)\, |\psi(1, 2)\rangle \end{aligned} \tag{9.51}$$

and this completes the proof. ▌

We need also another elementary theorem. Suppose we consider observables A and B which have non vanishing matrix elements between states belonging to V_Δ and V_{Δ^*}. Then we have the following theorem:

Theorem 9.6. For pairs of observables connecting the two manifolds V_Δ and V_{Δ^*}, in general, the joint probability of getting a pair of eigenvalues when the composite system is in state (9.48) does not factorize into the product of the probabilities of the single events.

Proof. This theorem is easily proved by considering the two elementary operators $A = |\varphi_r\rangle\langle\varphi_s| + |\varphi_s\rangle\langle\varphi_r|$ and $B = i[|\varphi_r\rangle\langle\varphi_s| - |\varphi_s\rangle\langle\varphi_r|]$ with $r \in \Delta$ and $s \in \Delta^*$. The eigenvectors of these operators are

$$A = \pm 1 \Rightarrow \frac{1}{\sqrt{2}}[|\varphi_r\rangle \pm |\varphi_s\rangle] \qquad B = \pm 1 \Rightarrow \frac{1}{\sqrt{2}}[|\varphi_r\rangle \pm i\,|\varphi_s\rangle] \tag{9.52}$$

Then it is trivial to see that if $|\Pi\rangle$ contains the state $|\varphi_r\rangle$ and $|\Phi\rangle$ contains the state $|\varphi_s\rangle$:

$$Pr(A = 1\&B = -1) \neq Pr(A = 1) \cdot Pr(B = -1) \tag{9.53}$$

with obvious meaning of the symbols. ∎

With these premises we can now be more precise about our line of thought. As we have discussed in great detail before, our definition of a non-entangled state of two identical particles uniquely identifies two orthogonal submanifolds V_Δ and V_{Δ^*}. Given a pair of observables A and B commuting with P_Δ and with Q_{Δ^*} respectively, let us consider their restrictions $\tilde{A}$ and $\tilde{B}$ to V_Δ and V_{Δ^*}. The first of the theorems we have proved implies then the factorizability of the joint probabilities of getting a pair of eigenvalues of $\tilde{A}$ and $\tilde{B}$. The second one shows that for operators connecting states of V_Δ and V_{Δ^*} this does not hold, in general.

Now we can tackle the problem of the local factorizability of probabilities. To fully understand it, it is sufficient to identify V_Δ and V_{Δ^*} with the Hilbert spaces spanned by vectors which, in the configuration representation, have compact support in the two disjoint subsets Δ and Δ^* of the real axis. In this case, if we identify V_δ and V_{δ^*} with the Hilbert spaces of the square integrable functions of compact support in the indicated intervals, our theorem tells us that the joint probability of finding one particle in the space interval δ and one in δ^* factorizes provided the wave function is non-entangled, with factors having the appropriate supports, one in Δ and one in Δ^*.

This makes precise that the non-entangled state (9.48) exhibits, in the considered case, local factorizability of position probabilities. It has however to be remarked that, according to our second theorem, if consideration is given to two observables A and B such that their matrix elements

$\langle x| A |x^*\rangle$ and $\langle x| B |x^*\rangle$ do not vanish for $x \in \Delta$ and $x^* \in \Delta^*$, then, in general, in the considered state:

$$Pr(A = a_i \& B = b_j) \neq Pr(A = a_i) \cdot Pr(B = b_j) \tag{9.54}$$

i.e., the joint probability of getting the outcomes a_i and b_j for such observables does not factorize. The situation should now be clear. The non-entangled character of the state $|\psi(1, 2)\rangle$ identifies precise manifolds and, correspondingly, observables commuting with the projection operators onto them. The restrictions of such observables to the considered manifolds identify the relevant properties related to the state. It is just the joint probabilities referring to such properties which factorize. We perfectly agree that the physically really interesting case is the one in which the manifolds V_Δ and V_{Δ^*}, as well as the submanifolds V_δ and V_{δ^*}, are associated to disjoint sets of position variables, in which case local factorizability holds in the situation described above. Actually as we have already done and as we are going to discuss in the next section we are inclined to attach a particularly relevant role to this case since we are convinced that position variables must be endowed with a priviliged status. However, from the point of view we have taken in this paper, which makes systematic reference to the problem of the legitimacy of attributing properties to the constituents, it seems perfectly appropriate and correct to adopt the definition we have chosen to identify non-entangled states of identical particles. In the general case, the probabilities which factorize are related to observables different from positions, and as such they are not local.

A final remark. All previous considerations can be easily generalized to non-entangled states of N identical fermions. In such a case, if one deals with the typical non-entangled state:

$$|\psi(1,\ldots, N)\rangle = \sqrt{\binom{N}{K}}\, P_A[|\Pi(1,\ldots, M)\rangle\, |\Phi(M+1,\ldots, N)\rangle] \tag{9.55}$$

and if one considers two arbitrary projection operators $P(1,\ldots, M)$ and $Q(M+1,\ldots, N)$, which project onto the two *one-particle orthogonal* submanifolds $V_\Delta^{(M)}$ and $V_{\Delta^*}^{(K)}$ containing $|\Pi\rangle$ and $|\Phi\rangle$, respectively, one can easily prove that

$$\begin{aligned}\langle\psi|\, P_A\tilde{S}[P(1,\ldots, M) \otimes Q(M+1,\ldots, N)]\, P_A\, |\psi\rangle \\ = \langle\psi|\, P_A\tilde{S}[P(1,\ldots, M) \otimes I(M+1,\ldots, N)]\, P_A\, |\psi\rangle \\ \cdot \langle\psi|\, P_A\tilde{S}[I(1,\ldots, M) \otimes Q(M+1,\ldots, N)]\, P_A\, |\psi\rangle\end{aligned} \tag{9.56}$$

which is the obvious generalization of our previous result (9.49) for two identical fermions. Moreover, a similar argument can be developed for the boson case.

9.3. Some Useful Remarks Concerning Almost Perfect Non-Entanglement

Having made precise the idea of a "group of particles" of a system of identical particles being non-entangled with the remaining ones (or, more precisely, the idea of complete set of properties objectively associated to the "subquasets" of a "quaset" of N identical particles) we can reconsider the situation we have envisaged at the beginning of this section, i.e., the case in which a Helium atom is here (at the origin O of our reference frame) and a Lithium atom is there, let us say at a distance d from O. Our worries concerned the legitimacy of claiming "there is a Helium atom at the origin" or "there is a Lithium atom at a distance d from the origin" when one takes into account, e.g., the identity of the electrons of the two systems which requires the state vector to be totally skew-symmetric under their exchange. To discuss the conceptually relevant aspects of this problem we will confine, for simplicity, our considerations only to the electrons which are involved, disregarding the nuclei of the atoms—and the necessary anti-symmetrization concerning the protons and the neutrons.

The puzzling question we have to face is the following: since the electrons are indistinguishable, in which sense can we state that two of them are around the origin (to make the Helium atom which is there) and three at a distance d? And then, in which sense can one claim that "there is a Helium atom at the origin?" The answer, as we have stressed in this section, can be only given by paying the due attention to the total state vector of the complete system "Helium + Lithium." Such a state vector has the form:

$$|\psi^{(5)}\rangle \propto \mathscr{G}[|\text{Helium}^{(2)}\text{ here}\rangle \otimes |\text{Lithium}^{(3)}\text{ there}\rangle] \tag{9.57}$$

And now we are in trouble. The factors $|\text{Helium}^{(2)}\text{ here}\rangle$ and $|\text{Lithium}^{(3)}$ there$\rangle$ of the state inside the square brackets at the r.h.s. do not satisfy exactly our fundamental request of being one-particle orthogonal.

However, we can explicitly evaluate integrals like the one of Eq. (9.15), which, when they vanish, make legitimate precise claims concerning the objective properties of the considered subquasets. We notice that the modulus of the relevant integral is of the order of the overlap integrals of the electronic wave functions. Since they decrease exponentially outside a region of 10^{-8} cm from the corresponding nuclei, we immediately see that, for a distance between the two nuclei of the order of 1 cm, the relevant integral turns out to have a value of the order of $10^{-10^{16}}$. It is easy to convince oneself that this figure represents also the probability that, if one has an apparatus devised to check whether there is an Helium atom around the origin, it will not detect such an atom.

So, our claim "there is a Helium atom around the origin" is, strictly speaking, not perfectly correct but has only an approximate validity. However, as appropriately pointed out by the authors of ref. 18, "*all assertions of physics have that kind of approximation. When we state that the heat passes spontaneously only from a hotter to a cooler body in contact, we really mean that in a real case it is extremely probable that it should do so.*"

Concluding, in the considered case the formal conditions which are necessary for attributing consistently objective properties to "a group of particles" are satisfied to an extremely high degree of accuracy so that, precisely in the same way as we consider valid all the (unavoidably approximate) assertions about physical systems, we can confidently say that "there is a Helium atom here *and* a Lithium atom there."

A concluding remark is appropriate. The analysis we have performed has played an important role in making clear what are the formal features which make legitimate, in a rigorous or in an extremely well approximate way, to consider, in the case of a system of identical constituents, two subgroups of them as disentangled from each other and as possessing objectively precise properties.

To fully appreciate the real relevance of our considerations we invite the reader to consider the case in which, in place of the state (9.57) one is dealing with a state like:

$$
\begin{aligned}
|\psi^{(5)}\rangle \propto \mathcal{G}\ [&|\mathrm{Helium}^{(2)}\ \mathrm{here}\rangle \otimes |\mathrm{Lithium}^{(3)}\ \mathrm{there}\rangle \\
&+ |\mathrm{Lithium}^{(3)}\ \mathrm{here}\rangle \otimes |\mathrm{Helium}^{(2)}\ \mathrm{there}\rangle] \qquad (9.58)
\end{aligned}
$$

which is perfectly possible and relatively easy to prepare, and which would not make legitimate, in any way whatsoever, to make claims about what is here being a Helium rather than a Lithium atom. We stress that the embarrassment with a state like the one we have just considered does not arise from the fact that the strict conditions which would make our claims absolutely rigorous are not exactly satisfied, but from the fact that the state of the whole system is genuinely entangled due to the fact that it is obtained by antisymmetrizing a nonfactorized state. Even if the states of the Helium and Lithium atoms would satisfy our strict requirements of one-particle orthogonality, no objective property referring to the region around the origin (and the one at a distance d from it) could be identified and claimed to be possessed.

Finally, we mention that, in the case of the state (9.57), the approximation of disregarding the overlap of the electronic (or nucleonic) wave functions associated to the two atoms, is practically equivalent to ignoring the request of totally antisymmetrizing the state vector under the exchange of the electrons of the Helium and Lithium atoms.[23] Once more the legitimacy of doing so can be explicitly evaluated by taking into account to which extent one can disregard (*de facto* they are absolutely negligible) the exchange effects. However, giving up the antisymmetrization request amounts to considering the electrons of the Helium as distinguishable from those of the Lithium atom. If one makes this step, then one sees that the conclusions we have drawn concerning systems of identical constituents reduce to those we have derived for the case in which they are distinguishable.

9.4. Completely Non-Entangled Indistinguishable Fermions

As in the case of N distinguishable particles, once one has identified two "groups" of particles which are non-entangled with each other, one can raise the question of whether also the"members" of each subset can be subdivided into non-entangled subsubsets. We will limit ourselves to give the definition and the associated theorem (which is easily proved) which characterizes the states corresponding to completely non-entangled identical fermions.

Definition 9.2. The pure state $|\psi(1,\ldots,N)\rangle \in \mathscr{H}_A^{(N)}$ describing a system of N indistinguishable fermion particles, is completely non-entangled if there exist N mutually orthogonal one-dimensional projection operators P_i, $i = 1 \cdots N$, such that:

$$Tr^{(1+\cdots+N)}[E_i \, |\psi(1,\ldots,N)\rangle\langle\psi(1,\ldots,N)|] = 1 \qquad \forall i = 1 \cdots N \tag{9.59}$$

where $E_i = I^{(1)} \otimes \cdots \otimes I^{(N)} - (I^{(1)} - P_i^{(1)}) \otimes \cdots \otimes (I^{(N)} - P_i^{(N)})$.

The quantity $Tr^{(1+\cdots+N)}[E_i \, |\psi^{(N)}\rangle\langle\psi^{(N)}|]$, where the projection operators E_i are totally symmetric under the exchange of two arbitrary particles, gives the probability of finding one fermion in a well definite one-dimensional manifold, the one onto which P_i projects. Therefore, in a completely non-entangled physical system composed of identical constituents,

[23] What we have in mind should become clear if one makes reference, e.g., to the evaluation of exchange effects using perturbation theory. In such a case, for example, the matrix elements of the Hamiltonian between states which are products of states with disjoint supports vanish, so that one can avoid performing the antisymmetrization procedure.

Eqs. (9.59) guarantee that all the N particles possess a complete set of objective properties.

The content of the following theorem, which is a generalization of the one we have already proved in the simpler case of two particles, shows the relevance of Definition 9.2:

Theorem 9.7. A system $\mathscr{S} = \mathscr{S}_1 + \cdots + \mathscr{S}_N$ of N identical half-integer spin particles described by the pure state $|\psi(1,\ldots,N)\rangle$ is completely non-entangled iff it can be obtained by antisymmetrizing a completely factorized state. Note that the factors can be assumed to be orthogonal among themselves without any loss of generality.

10. IDENTICAL BOSONS

We have now to face the problems of property attribution and entanglement in the case of a system of N indistinguishable bosons. The main difference with respect to the case of identical fermion systems derives from the fact that, when one splits the set of the N particles into two or more subsets which have a complete set of properties, it may happen that two such subsets containing the same number L of particles are associated to the same state $|\Gamma^{(L)}\rangle$. Alternatively, as we will see, the various subsets must be associated to states (which may very well contain a different number of particles) which are one-particle orthogonal among themselves in the precise sense defined in Section 9.1.1. These are the only two cases which can give rise to disentangled subsets of the whole set of particles.

For simplicity, we will confine our considerations to the possible occurrence of only two disentangled subsets (we shall suggest subsequently how one has to proceed in the general case) and we will distinguish the cases of one-particle orthogonal and identical factors of the factorized state we have to symmetrize.

Taking into account the analysis of Section 7.1.2 concerning the two-boson case, it is easy to see that the two above considered instances (i.e., the appearance of identical or one-particle orthogonal state vectors) cannot occur together if one requires the system to contain two subgroups possessing simultaneously a complete set of properties, i.e., which are non-entangled with each other.

10.1. Boson Subsets Corresponding to Different Properties

We recall that, in accordance with our Definition 8.1, in order to be allowed to speak of two non-entangled subsets of particles it must be possible to attach a complete set of properties to both subsets. As the reader can easily understand, the case we are interested in here—i.e., the one in which we exclude that the subsets have precisely the same properties—can be dealt with by repeating step by step the analysis we have performed for the fermion case. Accordingly, we will limit ourselves to recall the appropriate definitions and theorems, with the due changes, without going through detailed arguments and proofs.

In analogy with the fermion case, we will denote as $\mathscr{H}_S^{(R)}$ the Hilbert space of the state vectors which are totally symmetric for the exchange of all variables of R identical bosons. Moreover we define the projection operator P_S on the totally symmetric submanifold $\mathscr{H}_S^{(N)}$ of $\mathscr{H}^{(N)}$ as $P_S = \frac{1}{N!} S$, where S is the linear operator which acts in the following way on a N-single particle basis:

$$S\{|\varphi_{i_1}(1)\rangle \cdots |\varphi_{i_N}(N)\rangle\} = \sum_P P\{|\varphi_{i_1}(1)\rangle \cdots |\varphi_{i_N}(N)\rangle\} \tag{10.1}$$

where the sum is extended to all permutations P of the variables $(1,\ldots, N)$.

With reference to a state of $N = L + J$ such bosons, we consider a state $|\Gamma^{(L)}(1,\ldots, L)\rangle$ describing L such particles and, by the procedure of Section 9.1.1, we define the single particle manifolds $V^{\Gamma 1}$ and $V_\perp^{\Gamma 1}$ as well as the manifolds $V^{\Gamma L}$ and $V_\perp^{\Gamma J}$, with obvious meaning of the symbols. The very procedure to identify such manifolds guarantees that $|\Gamma^{(L)}(1,\ldots, L)\rangle \in V^{\Gamma L}$ and that the closed linear manifolds $V^{\Gamma L}$ and $V_\perp^{\Gamma J}$ are one-particle orthogonal.

It is easy to see that the properly normalized state vector $|\psi^{(N)}(1,\ldots, N)\rangle$ obtained by symmetrizing the direct product of $|\Gamma^{(L)}(1,\ldots, L)\rangle$ and an arbitrary vector $|\Lambda^{(J)}(1,\ldots, J)\rangle \in V_\perp^{\Gamma J}$ is:

$$|\psi^{(N)}(1,\ldots, N)\rangle = \sqrt{\binom{N}{L}}\, P_S[|\Gamma^{(L)}(1,\ldots, L)\rangle \otimes |\Lambda^{(J)}(L+1,\ldots, N)\rangle] \tag{10.2}$$

Once again, having identified the closed linear manifold $V_\perp^{\Gamma J}$ of $\mathscr{H}_S^{(J)}$, we can consider a complete orthonormal set spanning such a manifold, namely the set $\{|\Omega_{\perp j}^{(J)}(L+1,\ldots, N)\rangle\}$, and build the orthonormal set $\{|\epsilon_{\perp j}^{(N)}(1,\ldots, N)\rangle\}$ of states of $\mathscr{H}_S^{(N)}$:

$$|\epsilon_{\perp j}^{(N)}(1,\ldots, N)\rangle = \sqrt{\binom{N}{L}}\, P_S[|\Gamma^{(L)}(1,\ldots L)\rangle \otimes |\Omega_{\perp j}^{(J)}(L+1,\ldots, N)\rangle] \tag{10.3}$$

These states, which are the equivalent for the boson case of the ones we have introduced in Eq. (9.23), are properly normalized and mutually orthogonal and therefore the operator

$$E_{S\perp}^{\Gamma(N)} = \sum_j |\epsilon_{\perp j}^{(N)}(1,\ldots, N)\rangle\langle\epsilon_{\perp j}^{(N)}(1,\ldots, N)| \tag{10.4}$$

is a projection operator of $\mathscr{H}_S^{(N)}$.

It is now possible to characterize in a mathematically definite way, as we did when dealing with identical fermions, the fact that each of two subgroups of bosons possesses a complete set of different properties.

Definition 10.1. Given a system $S^{(N)}$ of N identical bosons in a pure state $|\psi^{(N)}\rangle$ of $\mathscr{H}_S^{(N)}$ we will claim that two subsets of cardinality L and J $(N = L + J)$, respectively, both possess a complete set of different properties iff there exists a state $|\Gamma^{(L)}\rangle$ of $\mathscr{H}_S^{(L)}$ such that

$$\langle\psi^{(N)}|\, E_{S\perp}^{\Gamma(N)}(1,\ldots, N)\, |\psi^{(N)}\rangle = 1 \tag{10.5}$$

where $E_{S\perp}^{\Pi(N)}(1,\ldots, N) = \sum_r |\epsilon_{\perp r}^{(N)}\rangle\langle\epsilon_{\perp r}^{(N)}|$ is the projection operator of $\mathscr{H}_S^{(N)}$ given by Eq. (10.4), which is uniquely identified, according to the previous procedure, by the assignment of the state $|\Gamma^{(L)}\rangle$.

From this definition one can easily derive the following remarkable results, whose proofs can be obtained by the same arguments leading to Theorems 9.3 and 9.4:

Theorem 10.1. A necessary and sufficient condition in order that a state $|\psi^{(N)}\rangle$ of the Hilbert space of N identical bosons allows the identification of two subsets of cardinality L and J $(N = L + J)$ of particles which possess a complete set of different properties is that $|\psi^{(N)}\rangle$ be obtained by symmetrizing and normalizing the direct product of two states $|\Gamma^{(L)}(1,\ldots, L)\rangle$ and $|\Delta^{(J)}(L+1,\ldots, N)\rangle$ of $\mathscr{H}_S^{(L)}$ and $\mathscr{H}_S^{(J)}$ respectively, where $|\Delta^{(J)}(L+1,\ldots, N)\rangle$ belongs to $V_{\perp}^{\Gamma L}$.

Theorem 10.2. Given a system $S^{(N)}$ of N identical bosons in a pure state $|\psi^{(N)}\rangle$ of $\mathscr{H}_S^{(N)}$ a sufficient condition in order that it contains two non-entangled subsets of cardinality L and J is that $|\psi^{(N)}\rangle$ can be written as:

$$|\psi^{(N)}\rangle = \sqrt{\binom{N}{L}}\, P_S[|\Gamma^{(L)}(1,\ldots, L)\rangle \otimes |\Delta^{(J)}(L+1,\ldots, N)\rangle] \tag{10.6}$$

where the states $|\Gamma^{(L)}(1,..., L)\rangle$ and $|\Delta^{(J)}(L+1,..., N)\rangle$ are "one-particle orthogonal" among themselves.

The previous definition and theorems yield clear and mathematically precise conditions under which it is possible to consider two subquasets of bosons as disentangled from each other. The condition of "one-particle orthogonality" is necessary in order to be allowed (i) to attribute a set of complete and different properties to each subgroup of the whole system and (ii) to extend the arguments of Section 9.2.4, concerning the possibility of *doing physics within each manifold by disregarding the other one*, also to the case of bosons. Thus, we have identified a first class of non-entangled boson states as the ones which are obtained by symmetrizing direct products of two "one-particle orthogonal" vectors.

10.2. Boson Subsets Corresponding to Identical Properties

Let us pass now to characterize the second class of N-boson states which, according to our general Definition 8.1, can be considered as non-entangled. In this case there is no need at all to resort to any complicated procedure to identify "one-particle orthogonal" manifolds; we can limit ourselves to claim that if the state $|\psi(1,..., N)\rangle$—N-even—is obtained by symmetrizing and normalizing a product state of two identical factors, i.e., if $|\psi(1,..., N)\rangle \propto S[|\Gamma(1,..., N/2)\rangle\, |\Gamma(N/2+1,..., N)\rangle]$, then it can be considered for sure as non-entangled. It is in fact apparent that we can attribute to both subgroups of $N/2$ particles the complete set of properties associated to the state $|\Gamma\rangle$.

Though mathematically clear, this situation may appear a bit problematic if one tries to develop considerations analogous to those of Section 9.2.4. In fact it is straightforward to show that now it is no longer possible to perform a physical measurement on a subgroup of $N/2$ particles without affecting, to some extent, the remaining ones. In this peculiar situation, in spite of the possibility of attributing a complete set of properties to both the component subgroups, it is practically impossible to devise any measurement process on one subgroup whose results will not depend on the presence of the other. We could say that there are correlations of a certain type which are intrinsically due to the fact that the identical particles are described by precisely the same state.

However, this fact does not give rise to any serious problem for two reasons. First of all, the "unavoidable correlations" we have just mentioned are related more to the fact that the subgroups are "truly identical —i.e., in precisely the same state" than to the Hilbert space description of the system. In a sense these effects are analogous to those one meets, even within a classical picture, when one compares the implications of Maxwell–Boltzmann statistics and those of Bose–Einstein statistics. Moreover, and much more important, it has to be stressed that since, as repeatedly remarked, the most interesting features of the entanglement and/or non-entanglement of identical constituents emerge in the case in which one has two subgroups confined within different spatial regions, the case of the product of two identical states has not a specific physical relevance.

In order to illustrate better this situation, let us resort to a simple physical example and let us consider a couple of spin zero particles described by the following state vector:

$$|\psi(1,2)\rangle = |\varphi_\Delta(1)\rangle\,|\varphi_\Delta(2)\rangle \tag{10.7}$$

where the normalized wave function associated to the ket $|\varphi_\Delta\rangle$ is defined on the (one-dimensional) real axis and has the following form:

$$\varphi_\Delta(x) = \langle x\,|\,\varphi_\Delta\rangle = \begin{cases} \dfrac{1}{\sqrt{\Delta}} & x \in \Delta \\[2ex] 0 & x \notin \Delta \end{cases} \tag{10.8}$$

It is our purpose to show that every conceivable position measurement we perform on a particle of the system, will unavoidably alter the whole wave function and therefore will modify the probabilistic predictions of subsequent measurements. Let us ask, for example, which is the probability of finding one of the two particles inside the closed interval $\Delta_1 \subset \Delta$ once precisely one particle has been found in a previous measurement to lie in the disjoint interval $\Delta_2 \subset \Delta$, and let us compare this result with the probability of finding precisely one particle within Δ_1 when no previous measurement has been performed. If we suppose to have found precisely one particle within Δ_2, the collapsed wave function is obtained by applying to the state (10.7) the usual operator $P_{\Delta_2} \otimes (I - P_{\Delta_2}) + (I - P_{\Delta_2}) \otimes P_{\Delta_2}$, where P_{Δ_2} projects onto the interval Δ_2, and then normalizing it:

$$|\tilde{\psi}(1,2)\rangle = \frac{1}{\sqrt{\dfrac{2\Delta_2}{\Delta}\left(1 - \dfrac{\Delta_2}{\Delta}\right)}}\left[|\tilde{\varphi}_{\Delta_2}\rangle\,|\varphi_\Delta\rangle + |\varphi_\Delta\rangle\,|\tilde{\varphi}_{\Delta_2}\rangle - 2\,|\tilde{\varphi}_{\Delta_2}\rangle\,|\tilde{\varphi}_{\Delta_2}\rangle\right] \tag{10.9}$$

In the previous equation the wave function $\tilde{\varphi}_{\Delta_2}(x)$ associated to the non-normalized state vector $|\tilde{\varphi}_{\Delta_2}\rangle$ has the following form:

$$\tilde{\varphi}_{\Delta_2}(x) = \langle x \,|\, \tilde{\varphi}_{\Delta_2} \rangle = \begin{cases} \dfrac{1}{\sqrt{\Delta}} & x \in \Delta_2 \\ 0 & x \notin \Delta_2 \end{cases} \tag{10.10}$$

In such a case the desired probability of finding the other particle within the interval $\Delta_1 \subset \Delta$ is easily obtained as:

$$\langle \tilde{\psi}(1,2)|\, P_{\Delta_1} \otimes (I - P_{\Delta_1}) + (I - P_{\Delta_1}) \otimes P_{\Delta_1} \,|\tilde{\psi}(1,2)\rangle = \frac{\Delta_1}{(\Delta - \Delta_2)} \tag{10.11}$$

On the contrary the probability of finding precisely one particle inside the interval Δ_1 when no previous measurement has been performed, is:

$$\langle \psi(1,2)|\, P_{\Delta_1} \otimes (I - P_{\Delta_1}) + (I - P_{\Delta_1}) \otimes P_{\Delta_1} \,|\psi(1,2)\rangle = \frac{2\Delta_1}{\Delta}\left(1 - \frac{\Delta_1}{\Delta}\right) \tag{10.12}$$

Since the two probabilities are clearly not equal for arbitrary choices of the disjoint intervals Δ_1 and Δ_2, one could be tempted to consider this fact as a manifestation of the outcome dependence which is typical of all entangled states. However, this argument is not correct since the strict and unavoidable correlations between position measurements, are simply due to the fact that the quantum particles are truly identical, and there is no need to invoke a special role played by their quantum nature. It is in fact possible to build up a very simple classical model, consisting of two particles which cannot be experimentally distinguished, which displays exactly the same correlated properties of our quantum pair.

In fact, let us consider a one-dimensional interval of length Δ and a couple of indistinguishable classical particles; assuming that each particle can, in principle, be found with the same probability in any finite subinterval of Δ of a given amplitude, we evaluate the probability distributions corresponding to the above quantum example, once we have randomly put the particles inside the interval. We find (not surprisingly) that the mere fact of having found precisely one particle inside, for example, the interval Δ_2, modifies the probability of finding the other particle within Δ_1, and this is due to the fact that the first information restricts the set of all possible ways of distributing the two particles within the interval. Moreover, it is easy to show that all the probability distributions of arbitrary position measurements coincide with those holding for the quantum case of two bosons in the same state.

Therefore, it is possible to interpret the peculiar correlations arising when dealing with bosons in the same state, as classical correlations which are simply due to the truly indistinguishable nature of the particles involved, the same situation occurring for a set of classical identical particles.

10.3. Some Remarks about Property Attribution

Contrary to what happens in the case of fermions, in which the possibility of attributing a complete set of properties to one subgroup automatically implies that also the remaining subgroup has a complete set of properties, for systems of identical bosons this is no longer true. This parallels strictly the situation we have already discussed in the case of two identical bosons when we have considered the state (7.14) obtained by symmetrizing the product of two non-orthogonal factors, and this is due to the fact that "one particle orthogonality," and not the standard orthogonality of the factors, is necessary to claim that both subgroups have different properties. To clarify once more this point, let us consider a state obtained by symmetrizing and normalizing two states which are not "one-particle orthogonal" and which contain an equal number of bosons:

$$|\psi^{(N)}(1,\ldots,N)\rangle = \mathcal{N}S[|\Gamma^{(N/2)}(1,\ldots,N/2)\rangle\, |\Lambda^{(N/2)}(N/2+1,\ldots,N)\rangle] \tag{10.13}$$

If one evaluates the scalar products of such a state with, e.g., the state $\mathcal{N}_\Gamma S[|\Gamma^{(N/2)}(1,\ldots,N/2)\rangle|\Gamma^{(N/2)}(N/2+1,\ldots,N)\rangle]$, or with $\mathcal{N}_\Lambda S[|\Lambda^{(N/2)}(1,\ldots,N/2)\rangle\, |\Lambda^{(N/2)}(N/2+1,\ldots,N)\rangle]$, one sees that such scalar products are, in general, not equal to zero even if the states $|\Gamma\rangle$ and $|\Lambda\rangle$ are orthogonal in the usual sense. This is sufficient to conclude that in a state like (10.13), in which $|\Lambda\rangle$ and $|\Gamma\rangle$ are not one-particle orthogonal, there is a nonzero probability of finding "two subgroups" in the same state. This in turn obviously implies that one cannot state that there are two subgroups possessing complete and different properties, while, in turn, the very fact that $|\Gamma^{N/2}(N/2+1,\ldots,N)\rangle \neq |\Lambda^{N/2}(N/2+1,\ldots,N)\rangle$ does not permit the claim that the two subgroups possess the same properties.

11. A COMMENT ON ENTANGLED ENTANGLEMENT

In this section we want to spend few words on a question addressed in ref. 22, i.e., whether the entanglement itself should be considered as an objective property of a given physical system. Since we have made sharply precise the distinction between entangled and non-entangled states, our answer is, obviously, affirmative. On the contrary, in the above paper the authors suggest that the non-separability property displayed by some quantum states "*is not independent of the measurement context,*" concluding that the entanglement is a contextual property. We would like, first of all, to stress that the real problem addressed by the authors is completely different from the one they seem to be interested in discussing, and that the conclusions they reach are absolutely obvious and rather trivial. What they actually discuss is whether subjecting a constituent of a many ($\geqslant 3$)-particle system to a measurement process one can leave the remaining particles in an entangled or non-entangled state, depending on the measurement one chooses to perform.

Their argument goes as follows. They consider a quantum state describing a three spin-1/2 particles, like the one considered by Greenberger, Horne and Zeilinger[23]:

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|z\uparrow\rangle_1\,|z\uparrow\rangle_2\,|z\uparrow\rangle_3 + |z\downarrow\rangle_1\,|z\downarrow\rangle_2\,|z\downarrow\rangle_3] \tag{11.1}$$

which is undoubtedly entangled. Then they remark that it is possible to leave the two non-measured particles in factorized or entangled states, depending on the different measurements which one chooses to perform on the third particle (the choice of measurement corresponding to what they call the *measurement context*). We remark that by measuring the spin of the third particle along the z-direction one leaves the remaining particles in a factorized state, while every other conceivable spin measurement leads to an entangled state. However, this conclusion has nothing to do with the objective entanglement of the initial state. It is absolutely obvious that the first two particles are sometimes left in an entangled state and sometimes not: this is a trivial consequence of the reduction postulate of standard quantum mechanics.

Actually, one can easily devise even more general (but always trivial) situations in which the fact that two particles of a three-particle system are entangled or not after the third has been subjected to a measurement may depend, not only on the measurement one performs, but even on the outcome one obtains. To this purpose let us consider the following entangled state of three distinguishable spin one-half particles:

$$|\psi\rangle = \frac{1}{\sqrt{3}}[|z\uparrow\rangle_1\,|z\uparrow\rangle_2\,|\omega_a\rangle_3 + (|z\uparrow\rangle_1\,|z\downarrow\rangle_2 + |z\downarrow\rangle_1\,|z\uparrow\rangle_2)|\omega_b\rangle_3] \tag{11.2}$$

where $|\omega_a\rangle$ and $|\omega_b\rangle$ are two eigenvectors belonging to different eigenvalues of an operator associated to an observable $\Omega^{(3)}$ of the third particle (which here, for simplicity we consider as distinguishable from the other two). It is evident that a measurement process of $\Omega^{(3)}$ performed on the third particle, produces a final state for the first two particles whose entanglement depends strictly on the measurement outcome: if the result $\Omega = \omega_a$ is obtained, particles 1 and 2 are described by a factorized state, while in case of $\Omega = \omega_b$ the two particles are left in an entangled state. Once more, this is due to the external intervention on the system and to wave function collapse, and therefore there is no need at all to attribute a special role to the measurement context to characterize the separability properties of a quantum system. The state after the measurement is, as always, completely different from the one before it.

PART IV: NON PURE STATES

This section is devoted to discuss briefly some problems which are relevant in connection with the locality issue and the validity of Bell's inequality. We will limit our considerations to the case of distinguishable particles.

12. CORRELATIONS AND BELL'S INEQUALITY

Let us consider, for simplicity, a system of two distinguishable particles in a non-pure state which is a statistical mixture, with weights p_j, of factorized states $|\varphi_j(1)\rangle\,|\theta_j(2)\rangle$. As implied by Theorem 4.3, for each of the states appearing in the mixture the expectation value of the direct product of two observables $A(1)$ of $\mathcal{H}_1$ and $B(2)$ of $\mathcal{H}_2$ also factorizes:

$$\begin{aligned}\langle\varphi_j(1)|\,\langle\theta_j(2)|\,A(1)\otimes B(2)\,|\varphi_j(1)\rangle\,|\theta_j(2)\rangle \\ = \langle\varphi_j(1)|\,A(1)\,|\varphi_j(1)\rangle\cdot\langle\theta_j(2)|\,B(2)\,|\theta_j(2)\rangle\end{aligned} \tag{12.1}$$

It follows that the expectation value of $A(1)\otimes B(2)$ in the non-pure state can be written as:

$$\langle A(1)\otimes B(2)\rangle = \sum_j p_j A_j B_j, \qquad p_i > 0, \qquad \sum_i p_i = 1 \tag{12.2}$$

where we have put $A_j = \langle\varphi_j(1)|\,A(1)\,|\varphi_j(1)\rangle$ and $B_j = \langle\theta_j(2)|\,B(2)\,|\theta_j(2)\rangle$.

We can now compare the expression (12.2) with the one giving the expectation value of the direct product of two observables in a hidden variable theory:

$$\langle A(1)\otimes B(2)\rangle = \int d\lambda\, \rho(\lambda)\, A(\lambda)\, B(\lambda), \qquad \rho(\lambda) > 0, \qquad \int d\lambda\, \rho(\lambda) = 1 \tag{12.3}$$

It is obvious that the two expressions (12.2) and (12.3) have exactly the same formal structure. Just as one can prove, starting from Eq. (12.3) and assuming that $|A(\lambda)|$ and $|B(\lambda)|$ are less than or equal to one, that Bell's inequality is satisfied, one can do the same starting from Eq. (12.2). The conclusion should be obvious, and it has been stressed for the first time in ref. 24: a non-pure state which is a statistical mixture of factorized states cannot lead to a violation of Bell's inequality.[24] The converse is obviously not true, once more for the simple reason that the correspondence between statistical ensembles and statistical operators is infinitely many to one. It could therefore easily happen that a non-pure state describing a statistical mixture of non-factorized (i.e., entangled) states is associated to the same statistical operator of a statistical mixture of factorized states. Since all expectation values depend only on the statistical operator, also in this last case one is lead to the same conclusion, i.e., that no violation of Bell's inequality can occur.

These considerations lead us to consider the relevant question of looking for mixtures which do not lead to violation of Bell's inequality, without worrying about their specific composition in pure subensembles. The appropriate formal approach is the following. Let us consider a given statistical ensemble E of a system of two particles and its statistical operator $\rho_E(1,2)$ and let us define an equivalence relation between ensembles in the following way:

$$[E^* \equiv E] \Leftrightarrow [\rho_{E^*}(1,2) = \rho_E(1,2)] \tag{12.4}$$

Our problem can now be reformulated in the following way: given a certain statistical operator $\rho(1,2)$ and considering the equivalence class of the statistical ensembles having it as its statistical operator, does this class contain at least one ensemble which is a statistical union of subensembles each of which is associated to a pure and factorized state? It is obvious that if one can answer in the affirmative to such a question, then one can guarantee that the considered statistical operator cannot lead to a violation of Bell's inequality. Some relevant investigations in this direction have appeared recently[27–29] but the general problem is extremely difficult and far from having found a satisfactory solution.

13. CONCLUSIONS

In this paper we have reviewed the peculiar features displayed by entangled quantum states, by analyzing separately the cases of two or

[24] We note that A. Shimony *et al.*[25] have claimed that no one had proved explicitly this fact before 1989 when it has been proved by Werner.[26] This is incorrect as one can check by reading ref. 24.

many distinguishable or identical quantum systems. We have given the appropriate definitions for the various cases of interest and we have derived the necessary and sufficient conditions which must be satisfied in order that a system can be considered entangled or non-entangled. The analysis has been quite exhaustive and, we hope, it has clarified some of the subtle questions about this extremely relevant trait of quantum mechanics.

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